GR

Incomplete notes by Ingemar Bengtsson, Spring 2004, covering the early part of the course only. The course is defined by B. Schutz: *A First Course in General Relativity*, Cambridge UP. For different perspectives see E. Schrödinger: *Spacetime Structure*, Cambridge UP (a bit old fashioned, but very clear), M. Ludvigsen: *General Relativity*, Cambridge UP and R. d'Inverno: *Introducing Einstein's Relativity*, Oxford UP, 1992. At a more elementary level there is the wonderful book by W. Rindler: *Relativity: Special, general and cosmological*, Oxford UP 2001. If you want to see the real thing, consult R. Wald: *General Relativity*, Chicago UP, 1984.

PREREQUISITES

What follows here is a summary of things that I expect you to know already, although the way I present it may be new to you.

Field theory

We begin in what I hope is a leisurely pace with a field theory much simpler than the one we will study, namely Maxwell's. As we will see, it is not all that different from Einstein's theory of the gravitational field if we think about it in the right way. Maxwell's theory describes the electromagnetic field through the equations

$$\begin{array}{l} \partial_\beta F^{\alpha\beta} = 4\pi J^\alpha \\ \\ \partial_\alpha F_{\beta\gamma} + \partial_\gamma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha} = 0 \end{array} \quad \Leftrightarrow \quad \Box A_\alpha - \partial_\alpha \partial \cdot A = -4\pi J_\alpha \ . \tag{1}$$

Here the electromagnetic field strength is

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha = \begin{bmatrix} 0 & -E_i \\ E_i & \epsilon_{ijk} B_k \end{bmatrix} \ . \tag{2}$$

At any given time then the field strength can be visualized as two vector fields **E** and **B** in space.

Concerning matter (which is not really part of the theory, it is just put in), it is described by a charge density and a current vector according to the scheme

$$J^\alpha = \begin{bmatrix} \rho \\ j_i \end{bmatrix} \ . \tag{3}$$

Maxwell's theory allows us to draw one important conclusion about matter, namely that

$$\partial_\alpha J^\alpha = \frac{1}{4\pi} \partial_\alpha \partial_\beta F^{\alpha\beta} = 0 \ , \tag{4}$$

as it must be because $F_{\alpha\beta} = -F_{\beta\alpha}$. The conclusion is that electric charge is conserved, by necessity.

How can we solve Maxwell's equations? One way to do it is to pick an arbitrary A_α, define the electromagnetic field strength as above, and then define

$$J^\alpha = \frac{1}{4\pi} \partial_\beta F^{\alpha\beta} \ . \tag{5}$$

This is known as "Synge's method" of generating solutions, and is evidently of no interest whatsoever.

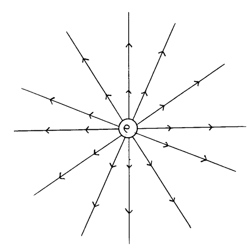
To get anything interesting out of the theory we must think of J^α as a function of some kind of variables—the position X^α of a point particle, some kind of charged matter field like Dirac's electron field ψ^a, and so on—that obey equations of motion or field equations of their own, possibly equations that again involve the electromagnetic field. So one possibility is

$$J^\alpha(x) = e \int d\tau \; \dot{X}^\alpha(\tau)\delta^{(4)}(x, X(\tau)) \tag{6}$$

together with Lorentz' equations for the trajectory of the charged particle. Once the problem is specified in this way the full set of equations goes non-linear and it becomes non-trivial to find solutions. But a long time has passed since the theory was first presented, and by now we know the key solutions. Here is a list:

1. Spherically symmetric solution. This solution describes a spherically symmetric blob of charged matter with $\rho(r > R) = 0$; in the vacuum outside there is a radially directed electric field. The solution is unique and static (independent of time). If $\rho = 0$ the solution becomes trivial. The strength of the electric field can be computed in an interesting way. Consider the total charge

$$Q = \int dV \; \rho = \frac{1}{4\pi}\int_V dV \; \partial_i E_i = \frac{1}{4\pi}\int_A dA_i E_i \; . \tag{7}$$



Now the total charges within two concentrical spheres with radii $> R$ are equal and the area of a sphere grows like r^2. Therefore we conclude that

$$E(r) \propto \frac{1}{r^2} \; . \tag{8}$$

This is known as a Coulomb field.

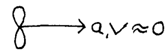
2. Propagating waves. If $J^\alpha = 0$ we have

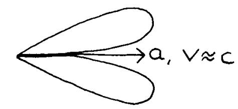
$$\Box F_{\alpha\beta} = \partial^\gamma \partial_\gamma F_{\alpha\beta} = -\partial^\gamma(\partial_\beta F_{\gamma\alpha} + \partial_\alpha F_{\beta\gamma}) = 0 \; . \tag{9}$$

Hence the components of the field strength obey the wave equation. Plane waves are a somewhat un–physical special case, but with suitable boundary conditions it is possible to get an arbitrary electromagnetic wave as a Fourier sum of plane waves.

3. Generation of waves. This is a bit more difficult. The point is that accelerated charges will emit waves. To see this, using a hand waving argument that is substantially correct, place a charge at the origin, move it "suddenly" (at $t = 0$) and then let it rest at a new position. Taking for granted that signals cannot propagate faster than light we can see what will happen: For $r > ct$ the field will be the Coulomb field centered at the original position, for $r << ct$ it will be a

Coulomb field centered at the new position. The field lines must be continuous, and if you try to match them together you will find a roughly spherical shell of transversely directed electric field lines. This is the pulse of radiation coming from the accelerated point charge.





Closer study reveals that electromagnetic radiation is always transverse, with its intensity at its greatest in a direction perpendicular to the direction of the acceleration as long as the velocity of the charge is small. When $v \approx c$ the radiation will be concentrated in the forwards direction because of a relativistic effect known from science fiction movies. If you have not seen enough of those, think of this as an exercise in spacetime geometry. (To be given explicitly later on.)

Returning to our spherical shell of radiation (at a distance r from the particle) we can estimate the strength of the electric field if we happen to know that the energy density in the electromagnetic field is

$$\mathcal{E} = \frac{1}{8\pi}(E^2 + B^2) \ . \tag{10}$$

As the shell grows its energy density must go down in such a way that the total energy in the shell stays constant. Hence $E \propto 1/r$. Alternatively, dimensional analysis plus the assumptions that E be linear in a and e gives

$$E \propto \frac{ea}{c^2}\frac{1}{r} \ . \tag{11}$$

The slow decrease with r, as compared to the Coulomb field, is an essential feature. Radio, not to mention astronomy, is a possibility only because of that.

I expect that you know Maxwell's theory well enough to recognize the story so far (except perhaps for the details of wave generation).

The purpose of this course is to bring your understanding of General Relativity Theory to about the same level. A curious remark that gives a hint of what is coming is the observation that there is a third player in Maxwell's theory, apart from fields and charges. It is the metric on Minkowski space:

$$\partial_\beta F^{\alpha\beta} \equiv \eta^{\alpha\gamma}\eta^{\beta\delta}\partial_\beta F_{\gamma\delta} \tag{12}$$

where

$$\eta^{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \ . \tag{13}$$

This is a piece of "background structure" that does not react to anything, even though it certainly acts on the field in some sense. You can think of GR as an attempt to redress this moral breaking of Newton's third law.

Maxwell's theory has a mathematical limit known as the Newtonian limit, in which the velocity of light is taken to infinity. All that then remains is a Coulomb force between pairs of electrically charged particles,

$$\text{Coulomb's law :} \qquad F_C = \frac{e_1 e_2}{r^2} \ . \tag{14}$$

On the face of it it looks very similar to

$$\text{Newton's law :} \qquad F_N = \frac{G m_1 m_2}{r^2} \tag{15}$$

where $G \approx 6.67 \cdot 10^{-11}\ m^3 kg^{-1} s^{-2}$ (it is not very well known!). There is a difference in strength between the two forces; between the electron and the proton in a hydrogen atom we find

$$\frac{F_C}{F_N} \approx 10^{40} \ . \tag{16}$$

Between the moon and the Earth I do not have an accurate figure, but it is clear that

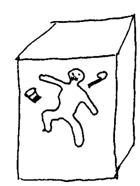
$$\frac{F_C}{F_N} \approx 0 \ . \tag{17}$$

Gravity always wins at large distances because electric forces tend to average each other out. This cannot happen for gravity, for reasons that make Coulomb's and Newton's laws very different from each other.

Set $F = ma$. The first particle has mass m_1 and charge e_1. Its acceleration according to Coulomb's law will be

$$a_1 = \frac{e_1 e_2}{m_1} \frac{1}{r^2} \ . \tag{18}$$

Newton's law gives



$$a_1 = G m_2 \frac{1}{r^2} \ . \tag{19}$$

The wonderful thing is that this is quite independent of the parameters describing the particle being accelerated.

If we use a suitably accelerated system of reference to describe the motion of the first particle then we will find that Newton's force disappears altogether—regardless of whatever mass or charge is carried by the particle whose motion we are tracking. This is sometimes known as the Principle of Equivalence, and dramatically illustrated by picturing Einstein trying the measure the gravitational force inside a freely falling elevator. If you have a good memory for electromagnetism you will recognize that there is some similarity to the notion of gauge transformations of the vector potential; the vector potential can not be measured because the transformation

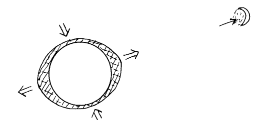
$$A_\alpha(x) \to A_\alpha(x) + \partial_\alpha \Lambda(x) \tag{20}$$

leaves the physics unchanged, even if the function Λ is chosen arbitrarily. So one can set an arbitrary component of A_α to zero.

We seem to be saying that the entire gravitational force can be transformed to zero. Indeed we are. This is true at any given point, but not over any larger region of space. Tidal forces are real.

It may be worthwhile to recall how tidal forces act. They are due to gradients in the gravitational force and hence their dependence on distance is

$$\text{tidal force} \sim \partial_r F_N \sim \frac{1}{r^3} \ . \tag{21}$$



The strong fall-off property means that the tidal force on the Earth is mostly due to the Moon, and not to the Sun. Anyway the effect of the tidal force is to squeeze a sphere into an ellipsoid. The solid earth resists this force better than the liquid oceans, and the result is tides—a measurable phenomenon quite independent of any systems of reference.

To complete the Newtonian theory of gravity we need an equation relating the gravitational potential to the matter density. Recall that the gravitational potential first enters when we write the equation of motion in the form

$$\ddot{x}_i = \partial_i \Phi(x) \ . \tag{22}$$

Because of the equivalence principle this equation is not directly relevant (unless we assume something rather specific about the reference system). Only relative accelerations between particles count. So, suppose a particle has the trajectory $x_i(t)$ and a nearby particle has the trajectory $x_i(t) + \xi(t)$. The relative acceleration is

$$\ddot{x}_i = \partial_i \Phi(x + \xi) - \partial_i \Phi(x) \approx \partial_i \partial_j \Phi(x) \xi_j \ . \tag{23}$$

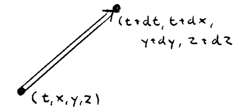
The equation that determines the gravitational potential is an equation for a particular combination of the physically relevant quantities $\partial_i \partial_j \Phi$, namely

$$\sum_{i=1}^{3} \partial_i \partial_i \Phi \equiv \triangle \Phi = 4\pi\rho \ , \tag{24}$$

where ρ denotes the mass density. This is Poisson's equation, and it completes the Newtonian theory. This particular way of looking at it will recur when we come to Einstein's equations later on. Eq. (23) will resurface as the non-relativistic analogy of something known as the geodesic deviation equation.

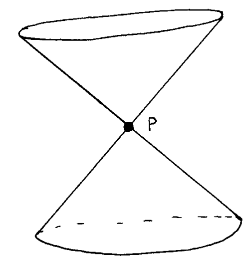
Minkowski space geometry

The point of the Special Relativity theory is not that spacetime is four dimensional. That was known long before H.G. Wells (1895). The point is that spacetime carries a natural metric. Think of spacetime as a vector space with an arbitrarily chosen origin. The "length squared" of a vector V^α is chosen to be



$$||V||^2 \equiv V^\alpha \eta_{\alpha\beta} V^\beta \quad \text{where} \quad \eta_{\alpha\beta} = \text{diag}(-c^2, 1, 1, 1) \tag{25}$$

and finally we set $c = 1$ for convenience. For a vector that begins at (t, x, y, z) and ends at $(t + dt, x + dx, y + dy, z + dz)$ we obtain



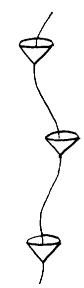
$$V^\alpha = \begin{bmatrix} dt \\ dx \\ dy \\ dz \end{bmatrix} \quad \Rightarrow \quad ds^2 \equiv ||V||^2 = -dt^2 + dx^2 + dy^2 + dz^2 \ . \tag{26}$$

In this (rather old fashioned) notation dx is the component of a vector that starts at x. It is not an infinitesimal object in any sense.

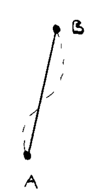
There are three possibilities:

$$\begin{array}{ll} ds^2 > 0 & \text{spacelike} \\ ds^2 < 0 & \text{timelike} \\ ds^2 = 0 & \text{lightlike (or null)} \end{array} \tag{27}$$

Depending on which case prevails the vector is said to be spacelike, timelike, or lightlike (null) and the points connected by the vector are said to be at spacelike, timelike, or lightlike separation. A curve is said to be spacelike, timelike or lightlike if its tangent vectors are always spacelike, always timelike, or always lightlike. (You can draw curves that are neither because ds^2 changes sign. But such curves turn out to be of no interest in relativity.) Each point serves as the tip of two cones, the forwards light cone and the backwards light cone, whose generators are the lightlike vectors emerging from that point. We note that a timelike curve always goes "into the lightcone".

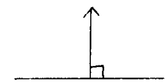


When drawing pictures of spacetime it is convenient to adopt the convention that lightlike vectors always have a slope of 45 degrees. Such "spacetime diagram" can be very useful, provided we remember that they distort the spacetime geometry in certain ways. Minkowski space geometry is very different from Euclidean geometry in some respects. For instance, every pair of points can be connected with a curve whose length is zero. However, if the separation between the points is timelike there will exist a longest timelike curve between them, if we use $d\tau^2 = -ds^2$ to measure length along the curve;

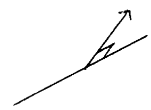


$$L = \int_{curve} d\tau \ . \tag{28}$$

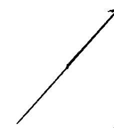
The notion of straight line does not really depend on the notion of length, and is certainly meaningful in Minkowski space.



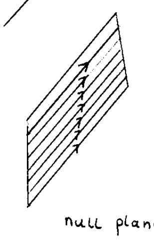
Some further terminology: A plane is said to be timelike if it contains two linearly independent lightlike vectors, and spacelike if it contains no such vector. Now let us look (in 2+1 dimensions for clarity) at a family of spacelike planes that are orthogonal to the timelike vector



$$v^\alpha = \begin{bmatrix} \cosh v \\ \sinh v \\ 0 \end{bmatrix} . \tag{29}$$

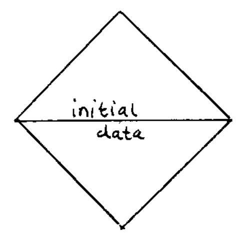


This vector is going lightlike as $v \to \infty$. (Do not bother about normalization, or multiply with a factor e^{-v} if it does bother you.) We can think of the planes as "planes of simultaneity" for a worldline with tangent vector v^α. The interesting thing, that one sees from a picture, is that when v reaches infinity the normal to the plane of simultaneity will lie in the plane itself. A plane is said to be lightlike precisely if it contains its own normal vector—and that will then be the only lightlike vector that it does contain.



Finally, some physical remarks:

- Photons, or electromagnetic signals, move along lightlike straight lines.
- Massive particles, people, and other creatures, move along timelike curves.
- Freely falling dittos move along timelike straight lines.
- All signals emerging from a point move on or within the forwards lightcone of that point.
- And this will be true in all field theories too.



In particular, in field theories we often pose the problem of computing the field in the future (or in the past), given its value (and that of its time derivative) at some given time. In Minkowski space we can pose initial values on any spacelike hypersurface, and compute the field within a region of spacetime with a lightlike boundary. But these things will be discussed in more detail later.

Tensors at a point

We now start a long mathematical detour aimed at giving us the tools to understand Einstein's General Relativity theory. All that we have when we start is a vector space $\mathbf{T}$, and vectors belonging to it:

$$\mathbf{u},\ \mathbf{v} \in \mathbf{T}\ ,\quad a,\ b \in \mathbf{R} \qquad \Rightarrow \qquad a\mathbf{u} + b\mathbf{v} \in \mathbf{T}\ . \tag{30}$$

The real numbers here are called scalars. The vector space has (many) bases such as $\{\mathbf{e}_i\}$ where $1 \leq i \leq n$. Using one of the bases any vector can be written in the form

$$\mathbf{u} = \sum_{i=1}^{n} u^i \mathbf{e}_i \equiv u^i \mathbf{e}_i\ , \tag{31}$$

where the last equality introduces Einstein's summation convention, according to which summation over pairs of repeated indices, one upstairs and one downstairs, are never written explicitly. The real numbers u^i are known as the components of the vector relative a given basis. It is common practice to denote the vector itself as u^i, with the choice of basis understood.

Now change the basis with an arbitrary linear transformation

$$\{\mathbf{e}_i\} \rightarrow \{\mathbf{f}_{i'}\}\ : \qquad \mathbf{f}_{i'} = \mathbf{e}_j M^j{}_{i'}\ . \tag{32}$$

The vectors themselves do not change when the basis is changing, so to compensate the change in basis the components must change too:

$$u^i \rightarrow u^{i'} = \Lambda^{i'}{}_j u^j \tag{33}$$

where Λ is some matrix. To see which matrix, note that

$$\mathbf{u} = u^i \mathbf{e}_i = u^{i'} \mathbf{f}_{i'} = \mathbf{e}_j M^j{}_{i'} \Lambda^{i'}{}_k u^k\ . \tag{34}$$

This has to work for all vectors, so it implies that

$$M^j{}_{i'} \Lambda^{i'}{}_k = \delta^j_k\ . \tag{35}$$

Hence M and Λ are each other's inverses.

Given a vector space $\mathbf{T}$ with basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ say, we can construct a new vector space $\mathbf{T} \otimes \mathbf{T}'$, where $\mathbf{T}'$ is a copy of $\mathbf{T}$ and the product vector space has the basis

$$\begin{array}{ccc} \mathbf{e}_1\mathbf{e'}_1 & \mathbf{e}_1\mathbf{e'}_2 & \mathbf{e}_1\mathbf{e'}_3 \\ \mathbf{e}_2\mathbf{e'}_1 & \mathbf{e}_2\mathbf{e'}_2 & \mathbf{e}_2\mathbf{e'}_3 \\ \mathbf{e}_3\mathbf{e'}_1 & \mathbf{e}_3\mathbf{e'}_2 & \mathbf{e}_3\mathbf{e'}_3 \end{array} . \tag{36}$$

It is clear that we can apply this idea to N-dimensional vector spaces and obtain N^2-dimensional and indeed N^n-dimensional vector spaces with n an arbitrary integer, by taking repeated tensor products. Such a vector space is known as the vector space of rank n tensors. The components of these vectors are given by t^{ij} for rank 2, t^{ijk} for rank 3, and so on. There will be natural occurring subspaces of symmetric and anti-symmetric tensors. The transformation properties of the components $t^{ijk\cdots}$ are defined by the transformation properties of t^i.

The next key idea is that given a vector space $\mathbf{T}$ we can consider the space of linear maps from $\mathbf{T}$ to $\mathbf{R}$ as a vector space $\mathbf{V}$ in its own right, with dimension equal to that of $\mathbf{T}$. That is to say,

$$\mathbf{v} \in \mathbf{V} \ \& \ \mathbf{u} \in \mathbf{T} \quad \Rightarrow \quad \mathbf{v}(\mathbf{u}) \in \mathbf{R} . \tag{37}$$

(Compare with "kets and bras" in quantum mechanics.) Given a basis $\mathbf{e}_i$ in $\mathbf{T}$ there exists a basis $\mathbf{w}^i$ in $\mathbf{V}$ defined by

$$\mathbf{w}^i(\mathbf{e}_j) = \delta^i_j . \tag{38}$$

An arbitrary vector $\mathbf{v} \in \mathbf{V}$ can be written as $\mathbf{v} = v_i\mathbf{w}^i$. This implies that

$$\mathbf{v}(\mathbf{u}) = v_i\mathbf{w}^i(u^j\mathbf{e}_j) = v_iu^j\mathbf{w}^i(\mathbf{e}_j) = v_iu^i \in \mathbf{R} . \tag{39}$$

We can repeat the tensor product construction starting from the *dual* vector space $\mathbf{V}$. This gives rise to tensors called *covariant tensors*, while the tensors arising from $\mathbf{T}$ are called *contravariant*. The transformation properties of the covariant tensors are given by the transformation properties of the contravariant ones.

	Components	Basis
Contravariant	$u^{i'} = \Lambda^{i'}{}_j u^j$	$\mathbf{e}_{i'} = \mathbf{e}_j \Lambda^{-1\ j}{}_{i'}$
Covariant	$v_{i'} = v_j \Lambda^{-1\ j}{}_{i'}$	$\mathbf{w}^{i'} = \Lambda^{i'}{}_j w^j$

The transformation rules are such that v_iu^i is an invariant scalar. Note that an expression like $\sum u^iu^i$ is not a scalar under general linear transformations. The rule is that you can contract one upstairs index with one downstairs, but in no other way.

Finally we come to an idea that comes as an "extra". The fact that it is an absolutely key ingredient in GR should not obscure the fact that tensor calculus as such does not need it. The idea is that a symmetric second rank tensor whose components g_{ij} form an invertible matrix can be used as a *metric tensor*, giving a *canonical identification* of the vector spaces $\mathbf{T}$ and $\mathbf{V}$:

$$u_i \equiv g_{ij} u^j \quad \text{are the components of a covariant vector.} \tag{40}$$

We will think of u_i as identical with u^i except that it has had its index lowered. The inverse of g_{ij} is denoted g^{ij} and can be used to raise indices in the same way.

Of course, the main reason why we want a metric tensor is that it enables us to define the *length* or *norm* of a vector through

$$||\mathbf{u}||^2 \equiv \mathbf{g}(\mathbf{u}, \mathbf{u}) \equiv g_{ij} u^i u^j \ . \tag{41}$$

In Euclidean geometry the metric tensor is simply a Kronecker delta; in fact then its presence in the equations is easily overlooked (and one tends to use abbreviations like $u^i \delta_{ij} u^j \to u^i u^i$, which is OK if we do rotations only, since they are linear transformations that preserve the form of δ_{ij}).

The main points of this—sketchy but nevertheless fairly complete—discussion are that

- So far, everything concerns vector spaces only.
- The metric enters at a late stage.

In this setting the metric is a rather harmless object that, following diagonalization and rescaling of the basis vectors, can be written in one of the forms $g_{ij} = \text{diag}(1,1,1,1)$, $g_{ij} = \text{diag}(-1,1,1,1)$, $g_{ij} = \text{diag}(1,-1,-1,-1)$, or $g_{ij} = \text{diag}(1,1,-1,-1)$. Only the number of minus signs matters. Note that this mathematical fact in a sense mirrors that of Einstein's equivalence principle.

Differentiable manifolds

Not to beat around the bush, let me give you the definition of a differentiable manifold as a quotation from Hawking and Ellis (*The Large Scale Structure of Space-Time*, a standard treatise from 1973):

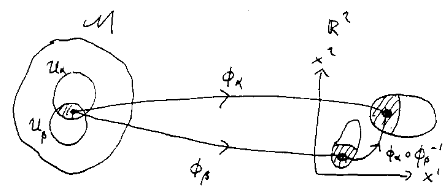
"A $\mathbf{C}^r$ n-dimensional manifold $\mathcal{M}$ is a set $\mathcal{M}$ together with a $\mathbf{C}^r$ atlas $\{\mathcal{U}_\alpha, \Phi_\alpha\}$, that is to say a collection of charts $(\mathcal{U}_\alpha, \Phi_\alpha)$, where the $\mathcal{U}_\alpha$ are subsets of $\mathcal{M}$ and the Φ_α are one-one maps of the corresponding $\mathcal{U}_\alpha$ to open sets in $\mathbf{R}^n$ such that

(1) the $\mathcal{U}_\alpha$ cover $\mathcal{M}$, i.e. $\mathcal{M} = \cup_\alpha \mathcal{U}_\alpha$,

(2) if $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ is non-empty, then the map

$$\Phi_\alpha \circ \Phi_\beta^{-1} \ : \ \Phi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \to \Phi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \tag{42}$$

is a $\mathbf{C}^r$ map of an open subset of $\mathbf{R}^n$ to an open subset of $\mathbf{R}^n$."



$\mathbf{C}^r$ means "has derivatives up to order r". Even so, at first sight the definition does not ring many bells with the uninitiated. However, standard treatises tend to be like this.

Actually it is all very simple and transparent. Let the manifold be $\mathbf{S}^2$, that is a two dimensional sphere. You can buy an atlas of this manifold in your local book store, and indeed it consists of a collection of charts, all of which define some map from a part $\mathcal{U}_\alpha$ of $\mathbf{S}^2$ to $\mathbf{R}^2$, i.e., to the flat pages. Inspection reveals that once you have a chart any point P in the manifold can be represented by its *coordinates*, namely the standard $\mathbf{R}^2$ coordinates (with origin in the lower left hand corner of the page, say) to which P has been mapped. Thus *the coordinates* (x^1, x^2) *are labels (names) for points.*

If you browse through your atlas (assumed to be a good one) you will find that for any point P on the sphere, there is a chart such that P appears on that chart. (Even for fairly obscure points on $\mathbf{S}^2$ such as those on Novaja Sembla.) This is condition (1) in the definition by Hawking and Ellis.

Some points, such as (probably) those on the Azores, occur on several charts. This brings us to condition (2) in Hawking and Ellis, which is there to ensure that the charts are "nice" and easy to work with. Technically, assume that the Azores figure on both page 28 and page 53, and let P be a point on one of these lovely islands. On page 28 we have

$$\begin{array}{l} x^1 = x^1(P) \\ x^2 = x^2(P) \end{array} \qquad \Leftrightarrow \qquad P = P(x^1, x^2) \tag{43}$$

(since the map Φ_{28} is one-to-one). On page 53 we have

$$\begin{array}{l} x^{1'} = x^{1'}(P) \\ x^{2'} = x^{2'}(P) \end{array} \qquad \Leftrightarrow \qquad P = P(x^{1'}, x^{2'}) \tag{44}$$

In the overlap region of the two charts (including the points on the Azores) we get

$$x^1 = x^1(P) = x^1(P(x^{1'}, x^{2'})) = x^1(x^{1'}, x^{2'}) \tag{45}$$

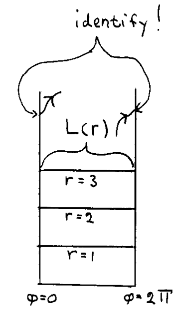
and similarly $x^2 = x^2(x^{1'}, x^{2'})$. Together these two functions define a function from $\mathbf{R}^2$ to $\mathbf{R}^2$, and condition (2) says that this function is r times differentiable. If this were not the case, you probably would not have bought the atlas in the first place.

We are now in a position where we can define differentiable functions from $\mathbf{S}^2$ to (say) the real numbers. This took some preparation because calculus is something that goes on in $\mathbf{R}^n$, so that *a priori* we could not do it. But now it is easy: A function $f: \mathbf{S}^2 \to \mathbf{R}$ is said to be differentiable if the function $F: \mathbf{R}^2 \to \mathbf{R}$ is differentiable in the standard sense, where F is defined by $F(x^1, x^2) = f(P(x^1, x^2))$. Here we are using a special chart that includes P. If P is on the Azores, or generally if P belongs to region where several charts overlap, then f can be represented by a quite different function $F': \mathbf{R}^2 \to \mathbf{R}$, viz.

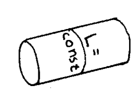
$$F'(x^{1'}, x^{2'}) = f(P(x^{1'}, x^{2'}) \ . \tag{46}$$

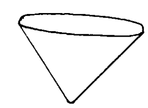
Precisely *because* of condition (2) the function F' is differentiable (r times) if and only if F is, so whether f is differentiable or not does not depend on the choice of chart. The atlas is consistent.

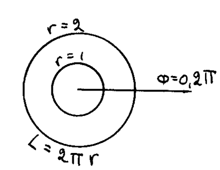
The space $\mathbf{R}^n$ can be covered with a single chart, while an atlas of $\mathbf{S}^2$ must contain at least 2 charts. The atlas you bought in the bookstore has more though, and indeed an atlas of $\mathbf{R}^n$ may also contain several charts (say, including polar coordinates).



The differentiability, or not, of a function is independent of the charts used, but it may depend on the atlas. An amusing fact is the following: We can ask whether it is possible to construct an atlas of $\mathbf{R}^n$, *not* including the standard (identity) chart, such that a function f that is *not* differentiable in the standard sense becomes differentiable with respect to the new atlas. The answer, given in the '80ies by Donaldson, is "no" if $n \neq 4$ but "yes" if $n = 4$. But you need not worry about this for the present.







What you should worry about is polar coordinates. In relativity we frequently know a spacetime *only* in some special chart, and we must be careful not to draw conclusions that in fact depend only on the peculiarities of the given chart. So suppose that we knew the plane only in the (r, ϕ)-chart. How would we recognize it? The chart in itself is always to be thought of as an open set (not including its boundaries), and in fact it does not cover quite all of the plane. Suppose we are told that, if a curve hits the boundary at $\phi = 2\pi$, it pops up again at $\phi = 0$, with the same value of r. This is beginning to look like a cylinder, or a cone, if we add points along the boundary of the chart and do the necessary identification—except that we do not know what to make of the boundary at $r = 0$. But suppose further that we are told about the length $L(r)$ of certain closed curves, defined by constant values of $r \neq 0$. What we are told is that $L(r) = 2\pi r$. Then we can see that our space is neither a cylinder nor a cone. It can be a plane (a special, smooth case of cone if you like). If we add a single point, at $r = 0$, to our space then the latter becomes the usual plane, at least in a topological sense—some further information about distances along curves of constant ϕ is needed to clinch matters. The lesson is that delicate attention to the details of what we see on the chart is needed, before we can say what the actual manifold is.

Tensors on differentiable manifolds

Recall that

- Vectors (and tensors) live in vector spaces.
- A differentiable manifold is typically not a vector space.

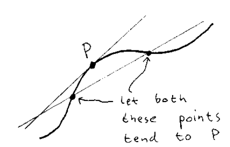
Add to this that

- We want to define vectors (and tensors) on differentiable manifolds.

How do we do it? The solution is to define a vector space **T**, called tangent space, at *every* point of the differentiable manifold. We can then construct tensor products $\mathbf{T} \otimes \mathbf{T}$, dual vector spaces **V**, etc., in an obvious way—again at each point separately.

The intuitive idea that defines tangent space is very simple. The actual definition used by mathematicians looks more scary at first sight. This happens because the mathematician (and the relativist) wants the definition to work *without reference to anything outside* the manifold itself—if we study a curved surface we do not necessarily regard it as "sitting inside" flat 3-space. This viewpoint will really pay off when we study curved spacetimes (such as our Universe) that indeed do not sit inside anything else.

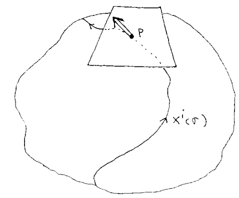
The intuitive idea is this: A tangent plane is a plane that just touches a curved surface at a given point. Suppose that the surface sits inside a flat 3-space to begin with. Any two points on a curve define a line. In the limit when these points coincide, this line tends to the tangent line of the curve, at the point. Similarly, three points on a surface define a plane, and when these points all tend to P, we get the tangent plane at P. All is well so far. But we want tangent vectors, of different lengths maybe, not just tangent spaces. Therefore we take "curve" to mean *parametrized curve* $x^i(\sigma)$; a curve is defined as a map from (a segment of) the real line to the manifold. If you like, we move along the curve with some velocity, "time" being measured by σ. We define the tangent vector of the curve $x^i(\sigma)$ at the point $x^i(0)$ as



$$v^i \equiv \frac{dx^i}{d\sigma}_{|\sigma=0} . \tag{47}$$

Now change the curve to $x^i(\sigma') = x^i(\frac{\sigma}{2})$, using the same functions x^i. It traces through the same points as the first curve, but it is parametrized in a different way. It has a tangent vector at $\sigma' = 0$ that differs from the original with a factor of 2:

$$u^i \equiv \frac{dx^i}{d\sigma'}_{|\sigma'=0} = \frac{d\sigma}{d\sigma'}\frac{dx^i}{d\sigma}_{|\sigma=0} = 2v^i \ . \tag{48}$$



A vector in a tangent plane will always be the tangent vector of *some* curve at the point P.

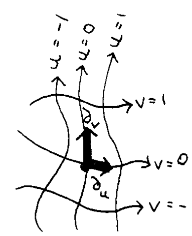
The generalization to higher dimensions is clearly trivial. But it is less trivial to drop all the references to the flat space in which the curved space is sitting. While our definition of tangent vectors is all right at is stands, a further step in an abstract direction will prove useful.

The key observation is this: There is a one-to-one correspondence between tangent vectors of curves on the one hand, and derivatives of functions in the direction of the curve on the other. With everything evaluated at P as usual:

$$\frac{df(x(\sigma))}{d\sigma}_{|\sigma=0} = \frac{dx^i}{d\sigma}\frac{\partial f}{\partial x^i}_{|\sigma=0} \equiv v^i\partial_i f \ . \tag{49}$$

We know about functions already, so we can *define* vectors as directional derivatives. Given a coordinate system, our definition allows us to express any tangent vector as

$$\mathbf{v} \equiv v^i\partial_i \ . \tag{50}$$



Note that we are relying on a special basis in tangent space, called the coordinate basis. Given coordinates (x^1, x^2, x^3), say, the basis consists of the three *vectors*

$$\partial_1 \equiv \frac{\partial}{\partial x^1} \qquad \partial_2 \equiv \frac{\partial}{\partial x^2} \qquad \partial_3 \equiv \frac{\partial}{\partial x^3} \ . \tag{51}$$

An arbitrary vector can be expanded as $\mathbf{v} = v^i\partial_i$. The basis vectors are in fact directional derivatives along a very special set of curves, namely the coordinate lines.

Our definition of tangent vectors $\mathbf{v} \in \mathbf{T}$ is nice when we want to see how components change if we change coordinates. The coordinate basis changes, while the vector itself is the same as ever. Suppose we have $\mathbf{v}$ expressed in the coordinate basis for coordinates x^i, and change to new coordinates $x^{i'} = x^{i'}(x)$. Clearly

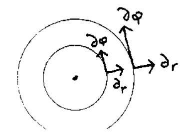
$$\mathbf{v} = v^i\partial_i = v^i\frac{\partial x^{j'}}{\partial x^i}\frac{\partial}{\partial x^{j'}} \equiv v^{i'}\partial_{i'} \ . \tag{52}$$

Hence—and this is a key formula—the new components are

$$v^{i'}(x') = \frac{\partial x^{i'}}{\partial x^j}v^j(x) \ . \tag{53}$$

As an example of this, let us transform from polar to Cartesian coordinates on the plane. So the new coordinates are given in terms of the old by $(x = r\cos\phi, y = r\sin\phi)$ and we obtain

$$\partial_\phi = \frac{\partial x}{\partial \phi}\partial_x + \frac{\partial y}{\partial \phi}\partial_y = -y\partial_x + x\partial_y \tag{54}$$



$$\partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y = \frac{x}{\sqrt{x^2+y^2}}\partial_x + \frac{y}{\sqrt{x^2+y^2}}\partial_y \ . \tag{55}$$

The components of the "old" basis vectors ∂_r and ∂_ϕ are thereby given in terms of the "new" basis vectors ∂_x and ∂_y.

It is important to observe that a much more interesting meaning can be given to the equations $x^{i'} = x^{i'}(x)$. Assume that we use the *same* coordinate system all the time. Let x^i be the coordinates of the point P, and assume that the point P itself is *moved* to the point P' whose coordinates, in the given coordinate system, are $x^{i'} = x^{i'}(x)$ (i.e. whose coordinates take these values). Going on in this vein, equation (53) then defines a tangent vector at P' as a function of a tangent vector at P; the catch phrase for this phenomenon is that "the (contravariant) vector moves with the map". This kind of quite general transformations of the manifold into itself used to be called *active coordinate transformations*; in most of the modern literature they are called *diffeomorphisms*. We will come back to them soon.

Having defined contravariant vectors on manifolds, as well as the tangent spaces $\mathbf{T}$ (one at each point), we can define tensors of arbitrary rank on manifolds, as vectors in the linear spaces $\mathbf{T}$, $\mathbf{T}\otimes\mathbf{T}$, $\mathbf{T}\otimes\mathbf{T}\otimes\mathbf{T}$, and so on (again at each point of the manifold). We can also define the dual space $\mathbf{V}$, the linear space of linear maps from $\mathbf{T}$ to $\mathbf{R}$, again at each point. It is known as *cotangent space.* Vectors in $\mathbf{V}$ are called covariant vectors or one-forms. In general a covariant tensor is a tensor whose components have indices downstairs. Given the coordinate basis for the tangent space, there is a natural basis in the cotangent space called dx^i,

$$dx^i(\partial_j) = \delta^i_j \ . \tag{56}$$

A warning: When I use the notation dx^i, it often denotes something completely different, namely the components of a tangent vector defined at the point whose coordinates are x^i.

Distances and metrics

The time has come to think of distances, at first in vector spaces, where the distance between the tip and the base point of a vector is called its norm. There are many ways to define that. If the components of $\mathbf{x}$ are $(x^1, x^2, \ldots, x^n)$ then the norm can be defined as

$$||\mathbf{x}|| = (|x^1|^p + |x^2|^p + \ldots + |x^n|^p)^{\frac{1}{p}} \tag{57}$$

for any integer p. Natural choices are $p = 2$, which leads to Pythagoras' theorem, and $p = 1$, which is useful if you are a taxi driver on Manhattan. In the nineteenth century Riemann realized that in differential geometry (that he was inventing) $p = 2$ wins hands down. Using a little more generality, he defined the length squared of a tangent vector as the scalar

$$||\mathbf{t}||^2 \equiv g_{ij} t^i t^j \ , \tag{58}$$

where $g_{ij}(x)$ is a symmetric non-degenerate (invertible) tensor defined at each point of the manifold.

In a vector space we can diagonalize a symmetric matrix, and rescale the basis, so that the metric takes the form

$$g_{ij} = \delta_{ij} \tag{59}$$

—and we are back to the previous $p = 2$ expression. (Actually, if some, let us say one, eigenvalue is negative this is not quite true. Then we get instead that $g_{ij} = \eta_{ij}$, where η_{ij} is the usual Minkowski space metric. Spaces for which this happens are called *Lorentzian*.) Minor *caveats* aside, on a general manifold it is still *not* true that we can find a smooth coordinate transformation such that $g_{ij}(x)$ takes a diagonal form at *every* point. That is a *very* interesting fact and leads to the theory of curved spaces.

About notation: Instead of eq. (58) one usually writes

$$ds^2 = g_{ij} dx^i dx^j \ , \tag{60}$$

and means by this exactly the same thing—the length squared of a tangent vector at the point x with components dx^i. There is a possible confusion here because in other contexts dx^i denotes a basis element in the cotangent space. Now one can interpret eq. (60) differently, as that operator acting on $\mathbf{T} \otimes \mathbf{T}$ which, when acting on $\mathbf{t} \otimes \mathbf{t}$, yields the length squared of the vector $\mathbf{t}$. But I do not think that one gains much compared to the old-fashioned interpretation of the formula that I tend to use.

Once we have defined the length of all tangent vectors there are many things we can do. First of all we can define the length of any curve. Let the curve be $x^i(\sigma)$ and let it connect the points P_1 (at $\sigma = \sigma_1$) to P_2 (at $\sigma = \sigma_2$). To get its length, we simply sum the lengths of its tangent vectors:

$$L = \int_{\sigma_1}^{\sigma_2} ds \equiv \int_{\sigma_1}^{\sigma_2} \sqrt{g_{ij} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma}} d\sigma \ . \tag{61}$$

Now you may (and should!) worry that the length depends on how the curve was parametrized—after all the lengths of the tangent vectors are affected by the parametrization. There is nothing to worry about though. Change the parametrization; let

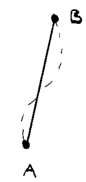
$$\sigma' = \sigma'(\sigma) \ . \tag{62}$$

Then $x^i(\sigma')$ and $x^i(\sigma)$ count as different curves, even though they pass through the same points in the manifold. Their tangent vectors will differ:

$$\frac{dx^i}{d\sigma'} = \frac{d\sigma}{d\sigma'}\frac{dx^i}{d\sigma} \ . \tag{63}$$

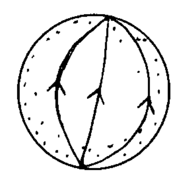
But the lengths of the curves agree:

$$\begin{aligned} L' \equiv \int_{P_1}^{P_2} \sqrt{g_{ij}\frac{dx^i}{d\sigma'}\frac{dx^j}{d\sigma'}}d\sigma' = \int_{P_1}^{P_2} \sqrt{g_{ij}\frac{dx^i}{d\sigma}\frac{dx^j}{d\sigma}\frac{d\sigma}{d\sigma'}}d\sigma' = \\ = \int_{P_1}^{P_2} \sqrt{g_{ij}\frac{dx^i}{d\sigma}\frac{dx^j}{d\sigma}}d\sigma \equiv L \ . \end{aligned} \tag{64}$$



So our definition of the length of a curve is a good definition.

Naturally we also want to know, given two points P_1 and P_2, which particular curve between them is the shortest, and what its length may be. The shortest curve is called a *geodesic*; later we will give a different but equivalent definition.



Some complications may occur: Given two points it is is not automatic that there exists a geodesic between them, or if it does that it is unique. An example of the latter difficulty occurs on the sphere, where the geodesics turn out to be arcs of Great Circles—and there are many of those connecting antipodal points on the sphere. In a different direction, in a spacetime (a Lorentzian manifold) there is *never* a shortest curve between two points, but there may be a *longest* timelike curve. (The alternative definition of geodesics, still waiting in the wings, will save the situation in general.) Since the tangent vector of a timelike curve has negative norm squared, its length is defined as

$$L = \int \sqrt{-g_{ij}\frac{dx^i}{d\sigma}\frac{dx^j}{d\sigma}}d\sigma \ . \tag{65}$$

Note the minus sign. With spacetimes in mind we modify the first definition of a geodesic slightly:

- A geodesic between two points P_1 and P_2 is that curve $x^i(\sigma)$ that extremizes the integral (61).

In spacetimes, if we consider timelike curves, we sneak a minus sign into the square root.

What does "extremize" mean? Quite generally, if $x(\sigma)$ is a function, with $\dot{x}$ denoting its derivative with respect to σ, and if L is a function of x and $\dot{x}$, then

$$S = \int_{\sigma_1}^{\sigma_2} L(x, \dot{x}) d\sigma \tag{66}$$

is a function of the function $x(\sigma)$. (L now stands for "Lagrange" rather than length.) The function*al* S has an extremum if small changes in $x(\sigma)$, that is $x(\sigma) \to x(\sigma) + \delta(\sigma)$, do not change the value of S.

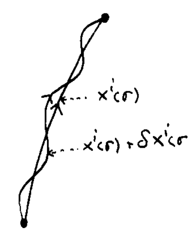
We have landed in a problem from analytical mechanics. A reminder: Let

$$L(x, \dot{x}) = \frac{m}{2}\dot{x}^2 - V(x) \ . \tag{67}$$

Then a small change in $x(\sigma)$ causes a small change in the *action* S, as follows:

$$\delta S = \int_{\sigma_1}^{\sigma_2} \delta L d\sigma = \int_{\sigma_1}^{\sigma_2} \left[m\dot{x}\delta\dot{x} - \frac{dV}{dx}\delta x \right] d\sigma \ . \tag{68}$$

Performing a partial integration—which is allowed since the endpoints are not varied, $\delta x(\sigma_1) = \delta x(\sigma_2) = 0$, with the form of δx otherwise arbitrary—we see that



$$\delta S = -\int_{\sigma_1}^{\sigma_2} \left[m\ddot{x} + \frac{dV}{dx} \right] \delta x d\sigma \ . \tag{69}$$

This will vanish only if the integrand is zero, which leads to the equation of motion for x. Solve that equation and you have the function $x(\sigma)$ that we asked for.

In our case we want to extremize the integral (61). This is easy—but not the first time you do it! You start out with

$$\delta I = \int_{\sigma_1}^{\sigma_2} \frac{1}{2} \frac{\delta(g_{ij}\dot{x}^i\dot{x}^j)}{\sqrt{g_{mn}\dot{x}^m\dot{x}^n}} d\sigma \ . \tag{70}$$

Then you use a trick. The parametrization of the curve is so far arbitrary. By changing σ, we can change the length $\sqrt{g_{mn}\dot{x}^m\dot{x}^n}$ of the tangent vector until it equals 1, everywhere along the curve. What this means is that the parameter measures the actual length of the curve—we are going to exploit the fact that the distance between two points depends on the path travelled between them, but not on the velocity with which the path is traversed, so that we can choose a standard speed. What this does for you is that you can forget about the denominator in (70). An equivalent procedure is to start out with the integral

$$I = \int_{\sigma_1}^{\sigma_2} \frac{1}{2} g_{ij}\dot{x}^i\dot{x}^j d\sigma \ , \tag{71}$$

keeping the extra constraint in mind. Anyway we do it, we get

$$\delta I = \int_{\sigma_1}^{\sigma_2} \left[g_{ij}\delta\dot{x}^i\delta\dot{x}^j + \frac{1}{2}\delta x^k \partial_k g_{ij}\dot{x}^i\dot{x}^j \right] d\sigma \ , \tag{72}$$

and after a partial integration and some tidying up we find that this vanishes if and only if

$$g_{ij}\ddot{x}^j + \frac{1}{2}(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk})\dot{x}^j\dot{x}^k = 0 \ . \tag{73}$$

This is the *geodesic equation.* It is very important, so we write it once again, now with one index raised:

$$\ddot{x}^i + \frac{1}{2}g^{im}(\partial_j g_{mk} + \partial_k g_{mj} - \partial_m g_{jk})\dot{x}^j\dot{x}^k = 0 \ . \tag{74}$$

It gives the extremal—usually the shortest—path between two points. Note that—as required by our derivation—the geodesic equation has the property that

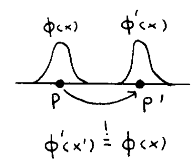
$$\dot{x}^2 \equiv \dot{x}^i g_{ij}\dot{x}^j = \text{constant} \ . \tag{75}$$

We can choose the parameter σ so that this constant equals one, which means that σ equals the arc length of the curve. In spacetimes, for timelike curves, there are minor sign changes in the derivation but the geodesic equation is the same. It now gives the longest curve between two points, parametrized by proper time.

Diffeomorphisms

We are interested in how things transform if we change coordinates, and even more interested in what happens if we move the points of spacetime around with an active diffeomorphism. Recall that, in a coordinate transformation, x and $x' = x'(x)$ are different coordinates for the same point, while, in a diffeomorphism, $x(P)$ and $x'(P')$ are coordinates for different points, and the points are transformed into each other according to $P \rightarrow P' : \ x' = x'(x)$. These are two different ideas expressed by the same formula. Here we take the second viewpoint, because it is of fundamental importance in Relativity theory. Indeed the name, "General Relativity", refers to the role played by diffeomorphisms in the foundations of the theory.

We transform not only points but also functions. We are only interested in *scalar* functions. They are defined by their behaviour under diffeomorphisms:



$$\phi(x) \rightarrow \phi'(x') = \phi(x) \ . \tag{76}$$

This is to say, given ϕ, there is a new function ϕ' that takes the same value at P' as ϕ did at P.

Vectors are objects that transform according to

$$V^{\alpha'}(x') = \frac{\partial x^{\alpha'}}{\partial x^\beta} V^\beta(x) \qquad \text{contravariant vector} \tag{77}$$

$$U_{\alpha'}(x') = \frac{\partial x^\beta}{\partial x^{\alpha'}} U_\beta(x) \qquad \text{covariant vector} \tag{78}$$

—and what is new, compared to what happens in vector spaces, is that vectors are moved by the diffeomorphism so that not only are their components shuffled around, they are also moved to a another tangent space, sitting over another point in the manifold. The main point is the same though, namely that $U_\alpha V^\alpha$ is a scalar function:

$$U_{\alpha'}V^{\alpha'}(x') = U_\beta \frac{\partial x^\beta}{\partial x^{\alpha'}} \frac{\partial x^{\alpha'}}{\partial x^\gamma} V^\gamma(x) = U_\beta \delta^\beta_\gamma V^\gamma(x) = U_\beta V^\beta(x) \ . \tag{79}$$

Covariant, contravariant, and mixed tensors of higher ranks transform in the way that should by now be obvious.

If we try to do calculus on the manifold, the first observation is that the gradient of a scalar behaves nicely. A simple calculation shows that it transforms like a covariant vector:

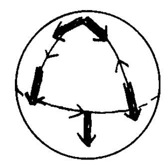
$$\partial_{\alpha'}\phi'(x') = \frac{\partial x^\beta}{\partial x^{\alpha'}} \partial_\beta \phi'(x'(x)) = \frac{\partial x^\beta}{\partial x^{\alpha'}} \partial_\beta \phi(x) \ . \tag{80}$$

But in the next step there is trouble: The "gradient" of a vector is not a tensor:

$$\begin{aligned} \partial_{\alpha'}U_{\beta'}(x') &= \frac{\partial x^\beta}{\partial x^{\alpha'}} \partial_\beta \left(\frac{\partial x^\gamma}{\partial x^{\beta'}} U_\gamma(x) \right) = \\ &= \frac{\partial x^\beta}{\partial x^{\alpha'}} \partial_\beta U_\gamma(x) + \frac{\partial x^\beta}{\partial x^{\alpha'}} \frac{\partial^2 x^\gamma}{\partial x^\beta \partial x^{\beta'}} U_\gamma(x) \ . \end{aligned} \tag{81}$$

The last term would not be present, did this object transform like a tensor. This is a serious problem because, if $T^{\alpha\beta}$ is a tensor, $T^{\alpha\beta}\partial_\alpha U_\beta$ is *not* a scalar. This makes the object $\partial_\alpha U_\beta$ an essentially useless one. (As an interesting side remark, you can easily check that the antisymmetric combination $\partial_\alpha U_\beta - \partial_\beta U_\alpha$ is a tensor. So there will be some things that one can usefully do without further definitions.)

As things stand then we do not have a useful notion of derivative. Another way to see that something is missing is this: We have defined tangent vectors, and tangent spaces, at each point separately. But this raises the question how we can recognize a "constant" tangent vector, or more generally how we can compare tangent vectors at *different* points—which is what we need to do when we take derivatives. In a flat space it is obvious how to move vectors around, but on a general manifold it is not obvious at all. If—although we really do not want to—we think of the vector at P as a vector in a flat embedding space containing our manifold, and if we use the flat space rules to move it to the new point P', then we find that the resulting vector does *not* lie in the tangent space at P'. We can repair this by brute force, if we project the vector at P' orthogonally down to the tangent space (relying on the metric on the flat embedding space to define orthogonal). In this way we have at least a definite procedure for how to move a vector from P to P', and we might tentatively say that two vectors at different points are equal if they can be transformed into each other in this way. But this would not work, because if we move the resulting vector at P' back to P using the same rules it will typically not agree with the original there. Nevertheless, we need a definition of "parallel transport of a vector on a manifold", and the definition that we will eventually adopt will be close to our first idea, with the crucial extra input that the projection down to the tangent space should be done "continuously" along some specified path connecting the two points. Precisely what vector one ends up with at P' will then depend on the path choosen. This is unavoidable (unless the manifold is flat), and in fact the precise way in which this happens will carry important information about the manifold itself.



Parallel transport

We will now define parallel transport of a vector properly. We begin with the notion of a *covariant derivative* of a vector. Our original problem was that $\partial_\alpha t^\beta$ does not transform like a tensor. So, we simple introduce an object $\Gamma^\gamma{}_{\alpha\beta}$ and call it the *affine connection.* (It will be called the "Christoffel symbol" soon, when we add an extra condition to make contact with our previous discussion.) Then we define the covariant derivative of a vector as

$$\nabla_\alpha t^\beta = \partial_\alpha t^\beta + \Gamma^\beta{}_{\alpha\gamma} t^\gamma \ . \tag{82}$$

By choosing the transformation properties of the affine connection suitably—*not* like a tensor !—one can ensure that the covariant derivative is a tensor. We can do this by hand, just so that any unwanted terms drop out from the transformation rules for $\nabla_\alpha t^\beta$. Note that, at this point in the story, the affine connection does not have any other properties, so it is not a uniquely defined object by any means. It is a new structure: We start in any given coordinate system on an n dimensional manifold, and choose (arbitrarily) n^3 functions to represent the connection. The tranformation rules will tell us what the connection becomes in any other coordinate system. Note also that another notation for derivatives is

$$\partial_\alpha t^\beta \equiv t^\beta{}_{,\alpha} \qquad \nabla_\alpha t^\beta \equiv t^\beta{}_{;\alpha} \ . \tag{83}$$

The notation I am using seems to be the winning one.

We will need the covariant derivative of every kind of tensor. By now there is no choice about this, if we insist on Leibnitz' rule for derivatives. The covariant derivative of the outer product of two vectors is

$$\nabla_\alpha (p^\beta q^\gamma) = \nabla_\alpha p^\beta q^\gamma + p^\beta \nabla_\alpha q^\gamma = \partial_\alpha (p^\beta q^\gamma) + \Gamma^\beta{}_{\alpha\delta} p^\delta q^\gamma + \Gamma^\gamma{}_{\alpha\delta} p^\beta q^\delta \ . \tag{84}$$

The covariant derivative of an arbitrary rank two contravariant tensor is therefore

$$\nabla_\alpha T^{\beta\gamma} = \partial_\alpha T^{\beta\gamma} + \Gamma^\beta{}_{\alpha\delta} T^{\delta\gamma} + \Gamma^\gamma{}_{\alpha\delta} T^{\beta\delta} \ . \tag{85}$$

The generalization to arbitrary rank should be obvious. If the indices are downstairs work things differently, but again the result is forced: We first note that

$$\nabla_\alpha \phi \equiv \partial_\alpha \phi \ . \tag{86}$$

The covariant derivative of a scalar is the ordinary derivative—it already transforms correctly. Now, given a covector p_α and a tangent vector q^α we know that $p_\alpha q^\alpha$ is a scalar. Therefore, again by Leibnitz' rule,

$$\nabla_\alpha (p_\beta q^\beta) = \nabla_\alpha p_\beta q^\beta + p_\beta \nabla_\alpha q^\beta \ . \tag{87}$$

But we have already agreed that the left hand side here can be written as an ordinary gradient. So, writing out the covariant derivative of the covariant vector and afterwards rearranging terms, we get

$$q^\beta (\nabla_\alpha p_\beta - \partial_\alpha p_\beta + \Gamma^\gamma{}_{\alpha\beta} p_\gamma) = 0 \ . \tag{88}$$

This must be true for all choices of q^β. Therefore

$$\nabla_\alpha p_\beta = \partial_\alpha p_\beta - \Gamma^\gamma{}_{\alpha\beta} p_\gamma \ . \tag{89}$$

One can check that this transforms like a covariant tensor. A little extra work like this, and we can write down the covariant derivative of an arbitrary tensor:

$$\begin{aligned} \nabla_\alpha T^{\beta\gamma\dots}{}_{\mu\nu\dots} = \partial_\alpha T^{\beta\gamma\dots}{}_{\mu\nu\dots} + \Gamma^\beta{}_{\alpha\delta} T^{\delta\gamma\dots}{}_{\mu\nu\dots} + \Gamma^\gamma{}_{\alpha\delta} T^{\beta\delta\dots}{}_{\mu\nu\dots} + \dots \\ -\Gamma^\sigma{}_{\alpha\mu} T^{\beta\gamma\dots}{}_{\sigma\nu\dots} - \Gamma^\sigma{}_{\alpha\nu} T^{\beta\gamma\dots}{}_{\mu\sigma\dots} - \dots \ . \end{aligned} \tag{90}$$

This much follows from our definition of $\nabla_\alpha t^\beta$ and Leibnitz' rule.

We can now *define* parallel transport of a vector along any given curve $x^\alpha(\sigma)$ by insisting that the covariant derivative evaluated in the direction of the tangent vector of the curve vanishes:

$$\dot{x}^\beta \nabla_\beta t^\alpha = 0 \tag{91}$$

or, written more explicitly

$$\frac{dx^\beta}{d\sigma}(\partial_\beta t^\alpha + \Gamma^\alpha{}_{\beta\gamma} t^\gamma) = \frac{dt^\alpha(x(\sigma))}{d\sigma} + \Gamma^\alpha{}_{\beta\gamma}(x(\sigma))\frac{dx^\beta}{d\sigma} t^\gamma(x(\sigma)) = 0 \ . \tag{92}$$

This is a set of coupled ordinary (as opposed to partial) differential equations and it follows that there is a unique solution for $t^\alpha(x(\sigma))$. That is to say,

- Any affine connection gives a unique prescription for parallel transport of a vector along a specified path.

In practice it may be hard to solve eq. (92), but that is another thing.

But what kind of an object *is* the affine connection? So far its only property is that it transforms in a funny way. Let us add one extra condition:

$$[\nabla_\alpha, \nabla_\beta]\phi = 0 \ . \tag{93}$$

An explicit calculation reveals that this is a non-trivial statement because the gradient of a scalar is a vector. In effect

$$\nabla_\alpha \nabla_\beta \phi = \partial_\alpha \partial_\beta \phi - \Gamma^\gamma{}_{\alpha\beta}\partial_\gamma \phi \ . \tag{94}$$

To make the commutator vanish it will be necessary and sufficient to set

$$\Gamma^\gamma{}_{\alpha\beta} = \Gamma^\gamma{}_{\beta\alpha} \ . \tag{95}$$

From now on we assume this to hold. To make sure that we can so assume, we must check that this condition holds in all coordinate systems if it holds in one. This is not obvious because the affine connection is not a tensor, but once the transformation rules for $\Gamma^\gamma{}_{\alpha\beta}$ are given explicitly, it is easily checked.

We will add another and more interesting condition, that will eventually enable us to tie the affine connection to our intuitive understanding of what parallel transport of vectors around curves in a curved space ought to mean. We require that the *length* of a tangent vector does not change under parallel transport. In equations, we require that

$$\nabla_\alpha t^\beta = 0 \qquad \Rightarrow \qquad \nabla_\alpha(t^\beta g_{\beta\gamma} t^\gamma) = 0 \ . \tag{96}$$

If we write this out and insist that it holds for arbitrary tangent vectors, we deduce that

$$\nabla_\alpha g_{\beta\gamma} = \partial_\alpha g_{\beta\gamma} - \Gamma^\delta{}_{\alpha\beta} g_{\delta\gamma} - \Gamma^\delta{}_{\alpha\gamma} g_{\beta\delta} = 0 \ . \tag{97}$$

Remarkably, eqs. (95) and (97) can be solved explicitly for the connection. This is a good exercise, and the result is

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}(g_{\delta\beta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta}) \ . \tag{98}$$

We learn that given a metric tensor, there is a preferred affine connection, and hence a preferred notion of parallel transport, available to us.

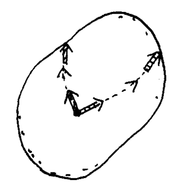
To summarize, the following conditions determine the covariant derivative uniquely:

i): $\nabla_\alpha \phi = \partial_\alpha \phi$
ii): $\nabla_\alpha t^\beta \equiv \partial_\alpha t^\beta + \Gamma^\beta{}_{\alpha\gamma} t^\gamma$ transforms like a tensor
iii): Leibnitz' rule
iv): $\Gamma^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\gamma\beta}$
v): $\nabla_\alpha g_{\beta\gamma} = 0$.

An affine connection that obeys conditions iv) and v) is often called a *Christoffel symbol*; it can be solved for in terms of the metric tensor.

One more thing, with or without condition v): Choose any tangent vector t^α at some point P. There will be many curves $x^\alpha(\sigma)$ whose tangent vectors at P equal the given vector;

$$\frac{dx^\alpha}{d\sigma}_{|\text{at } P} = t^\alpha \ . \tag{99}$$



Now we can ask, is there a curve $x^\alpha(\sigma)$ such that the above is true, and such that if we parallel transport t^α along the curve then the vector that results will agree with the tangent vector at every point along the curve? The question leads to an equation:

$$\frac{dx^\beta}{d\sigma}\nabla_\beta \frac{dx^\alpha}{d\sigma} = \frac{d^2x^\alpha}{d\sigma^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\sigma}\frac{dx^\gamma}{d\sigma} = 0 \ , \tag{100}$$

with the initial condition (99). This is a second order ordinary differential equation for $x^\alpha(\sigma)$, and it always has a solution (for some range of σ). Such a curve is, as it were, the straigthest possible curve through P in the direction of t^α, and is called a geodesic with respect to the affine connection. If the latter is the Christoffel symbol—as, from now on, we always assume—then it is also a geodesic in the sense of being a curve with extremal length, the way we defined geodesics earlier on. See eq. (74) for this. Evidently

$$\dot{x}^\beta \nabla_\beta \dot{x}^\alpha = 0 \tag{101}$$

is a particularly memorable form of the geodesic equation.

The Riemann tensor

The Riemann tensor is an object that helps to quantify the path dependence of parallel transport. It first turns up in the equation

$$[\nabla_\gamma, \nabla_\delta] t^\alpha = R^\alpha{}_{\beta\gamma\delta} t^\beta \ . \tag{102}$$

The left hand side is a tensor depending linearly on t^α, so therefore $R^\alpha{}_{\beta\gamma\delta}$ is a tensor, too. We have assumed that the commutator of two covariant derivatives vanish when acting on a scalar, but this time we get a non-zero result. An explicit calculation—that you *must* do yourself—shows that

$$R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha{}_{\delta\beta} - \partial_\delta \Gamma^\alpha{}_{\gamma\beta} + \Gamma^\alpha{}_{\gamma\mu}\Gamma^\mu{}_{\delta\beta} - \Gamma^\alpha{}_{\delta\mu}\Gamma^\mu{}_{\gamma\beta} \ . \tag{103}$$

Evidently, the equation that you should commit to memory is (102), not (103). Since $\Gamma = \Gamma(g, \partial g)$ we have $R = R(g, \partial g, \partial\partial g)$, or in words this is a function of the metric and its first and second derivatives (and it is linear in the second derivatives).

Directly from the definition one can show

$$R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu} = -R_{\beta\alpha\mu\nu} \tag{104}$$

Proof: The first is obvious, for the second consider $[\nabla_\gamma, \nabla_\delta]g_{\alpha\beta} = 0$.

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta} \tag{105}$$

Proof: Follows from the previous and the following equation.

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0 \tag{106}$$

Proof: Consider the Jacobi identity $[[\nabla_\beta, \nabla_\mu], \nabla_\nu]\phi = 0$.

$$\nabla_\gamma R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\gamma\mu} + \nabla_\mu R_{\alpha\beta\nu\gamma} = 0 \ . \tag{107}$$

Proof: Consider the Jacobi identity but with a vector for the derivatives to act on.

The first three equations imply that the number of algebraically independent components of the Riemann tensor is one (in two dimensions), six (in three), or twenty (in four dimensions). The reason this happens is as follows: Let the dimension be n. Because of eq. (104) we can think of the Riemann tensor as an $n(n-1)/2 \times n(n-1)/2$ matrix, since each anti-symmetric index pair can assume $n(n-1)/2$ different pairs of values. Because of eq. (105) it is a symmetric matrix. The number of independent components is now 1, 6, and 21 for $n = 2$, $n = 3$ and $n = 4$, respectively. Eq. (106) turns out to be empty for $n = 2, 3$ and gives one extra condition for $n = 4$. The fourth of the identities obeyed by the Riemann tensor is known as the *Bianchi identity*.

We can perform two index contractions. The tensor

$$R_{\mu\nu} \equiv R^\alpha{}_{\mu\alpha\nu} \tag{108}$$

is necessarily symmetric, and is known as the *Ricci tensor*. The scalar

$$R \equiv R^\mu{}_\mu \tag{109}$$

is known as the *curvature scalar*. Another tensor of interest is the *Einstein tensor*

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \ . \tag{110}$$

This particular combination is important because the Bianchi identities imply that

$$\nabla_\mu G^\mu{}_\nu = 0 \ . \tag{111}$$

It gets its name because it occurs in Einstein's field equations (as we will see).

Two dimensional spaces

What you need now is practical experience with Riemann tensors, and what they tell us, on some simple spaces. The simplest examples are of course flat spaces, where the metric tensor can be chosen to be independent of the cooordinates. Then the Christoffel symbols are zero, and so is the Riemann tensor. Famously the converse conclusion can be drawn—it the Riemann tensor is zero then the space is flat. Now let us consider a family of two dimensional spaces, described using coordinates (r, ϕ), and set

$$ds^2 = dr^2 + f^2(r)d\phi^2 \qquad \Leftrightarrow \qquad g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & f^2(r) \end{pmatrix} \ . \tag{112}$$

At the outset we do not assume anything about the function $f(r)$ (except differentiability), nor about the range of the coordinates. We want to compute the Riemann tensor for such spaces. There are many tricks (and computer programs) for doing this, but on this occasion you should do it in an entirely straightforward way.

You will find that the only non-vanishing Christoffel symbols are

$$\Gamma^r{}_{\phi\phi} = -ff' \qquad \Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{f'}{f} \ , \tag{113}$$

where f' is the derivative of f with respect to r. Up to index permutations, the only non-vanishing component of the Riemann tensor is

$$R^r{}_{\phi r\phi} = -ff'' \ . \tag{114}$$

The non-vanishing components of the Ricci tensor are

$$R_{rr} = -\frac{f''}{f} \qquad R_{\phi\phi} = -ff'' \tag{115}$$

and finally the curvature scalar is

$$R = -2\frac{f''}{f} \; . \tag{116}$$

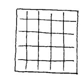
Actually, in two dimensions—and only in two dimensions, where there is only one independent component of the Riemann tensor—we can express the full Riemann tensor as a function of R:

$$R^{\alpha\beta}{}_{\gamma\delta} = \frac{1}{2}R(\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta} - \delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma}) \; . \tag{117}$$

(The right hand side has the right index symmetries and you can check that $R^{\alpha\beta}{}_{\alpha\beta} = R$.) In two dimensions, all the information about curvature is in one function, the scalar curvature.

Now let us look at some interesting special cases, namely those where R is constant. First, suppose $R = 0$. Then $f'' = 0$, and there are essentially only two solutions:

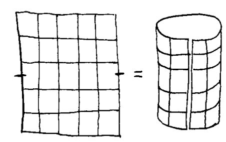
$$f(r) = r \quad \text{or} \quad f(r) = 1 \qquad \Rightarrow \qquad R = 0 \; . \tag{118}$$



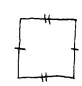
If $f = 1$ we have

$$ds^2 = dr^2 + d\phi^2 \; . \tag{119}$$

If we choose the coordinate ranges $-\infty < r < \infty$, $-\infty < \phi < \infty$ this is a flat plane expressed in Cartesian coordinates. But we do have other options. We can set $-\infty < r < \infty$ and $0 \leq \phi < 2\pi$ and then assume periodicity in ϕ. We get the *same* metric, but we are looking at a cylinder. A cylinder is intrinsically flat, basically because you can wrap a flat paper around it without wrinkling the paper. No local measurements of angles and lengths can distinguish between a cylinder and a plane—although global measurements can do it, like finding that there are closed geodesics (going round the cylinder).

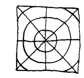


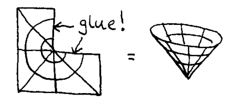
If you are disturbed by the fact that our formalism does not distinguish (locally) between two objects that are in fact different, then you have missed the point: The formalism as developed so far makes use only of the intrinsic geometry. It makes no reference to the way our spaces are sitting inside some other (say, flat) space. And, apart from global properties, that is the only way in which the cylinder and the plane differ. There do exist tools to describe this difference (by studying the behaviour of the normal vector of the surface), but here we leave this aside. When we do adopt the intrinsic viewpoint, we can in fact set $0 \leq r < 2\pi$, $0 \leq \phi < 2\pi$ and periodic in both coordinates. This is a flat torus. It cannot be made from flat paper because it cannot be embedded in a three dimensional flat space. But it is a perfectly well defined flat manifold anyway.



Now suppose $f(r) = r$. Then

$$ds^2 = dr^2 + r^2 d\phi^2 \; . \tag{120}$$



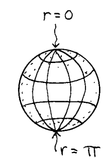


If we set $0 < r < \infty$, $0 \leq \phi < 2\pi$ and then assume periodicity in ϕ this is the flat plane in polar coordinates. (And it makes no sense to consider $-\infty < r < \infty$, basically because the expression for the metric misbehaves at $r = 0$.) But there are other possibilities. If we set $0 < r < \infty$, $0 \leq \phi < \frac{3\pi}{2}$ and then assume periodicity in ϕ this is in fact a cone, which is again flat. (Can be made from flat paper without wrinkling, that is, without disturbing the local geometry.) This time, if we try to add a point at $r = 0$, to complete the description given by our coordinate system, we find that the manifold becomes somewhat singular at that point, which is of course the tip of the cone.

Next we turn to the case that R is constant and positive, say

$$R = 2 \qquad \Rightarrow \qquad f'' = -f \ . \tag{121}$$

With the standard solution $f(r) = \sin r$ we get



$$ds^2 = dr^2 + \sin^2 r d\phi^2 \ . \tag{122}$$

If we set $0 < r < \pi$, $0 \leq \phi < 2\pi$ and then assume periodicity in ϕ this is in fact a round sphere of unit radius. If you do not recognize it, change the name of r and call it θ. If your recollection of spherical polar coordinates does not help, then you can always parametrize the surface $X^2 + Y^2 + Z^2 = 1$ as

$$X = \cos\phi \sin r \qquad Y = \sin\phi \sin r \qquad Z = \cos r \ , \tag{123}$$

with r and ϕ in the ranges specified, and work out the metric on the sphere as a quadratic form in $(dr, d\phi)$ starting from

$$ds^2 = dX^2 + dY^2 + dZ^2 \ . \tag{124}$$

That will reproduce eq. (122).

Note that closed geodesics (i.e., circles) of constant r shrink to zero length as we approach $r = 0$ or $r = \pi$. We cannot extend the range of r beyond these points. The coordinate system misbehaves there—we can change to other coordinates and use them to complete the manifold by adding single points there, but we cannot do better than that.

Next we take $R = -2$, that is, constant negative curvature. The equation becomes $f'' = f$, and the standard solution gives the metric as

$$ds^2 = dr^2 + \sinh^2 r d\phi^2 \ . \tag{125}$$

The "natural" coordinate range is $0 < r < \infty$, $0 \leq \phi < 2\pi$ and periodic in ϕ. To get to grips with what this means, we observe that—regardless of the sign in $R = \pm 2$—we can expand in a power series to investigate what goes on close to $r = 0$:

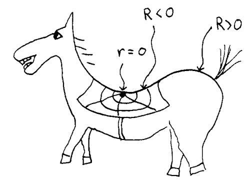
$$ds^2 = dr^2 + \left(r \mp \frac{r^3}{3!} + \ldots\right)^2 d\phi^2 \approx dr^2 + r^2 d\phi^2 \ . \tag{126}$$

Close to the point—indeed, close to any point—the metric is close to flat. So I will rely on what I know about flat space. All Riemannian spaces are "locally flat" to lowest order in an approximation scheme. This is the first observation, and it forces us to fix $0 \leq \phi < 2\pi$ and periodic in order to ensure that the space is locally smooth, regardless of the value of R.

We can draw a picture of the sphere ($R = 2$). Why not a picture of $R = -2$? That is harder, because it is impossible to embed the entire constant negative curvature plane in a 3-dimensional flat space. We can at most do a piece of it. To see what goes on, consider a circle at constant distance r_0 from the origin. Its circumference C is

$$C = \oint_{r=r_0} ds = \int_0^{2\pi} f(r_0) d\phi = 2\pi f(r_0) \ . \tag{127}$$

For the sphere we get $C = 2\pi \sin r_0 < 2\pi r_0$ while for constant negative curvature we get $C = 2\pi \sinh r_0 > 2\pi r_0$. So the circumference of a circle grows much quicker with radius if the curvature is negative than in does in flat space. In a sense a space with negative curvature has "more space" than a flat space (and a sphere has less!).



Surfaces that have negative, but varying, curvature in some region are easy to find. A standard example is a saddle on a horse, or the innner ring on a torus embedded in flat space. Indeed by now you should have no difficulty in visualizing a region around the origin for any choice of the function $f(r)$ in eq. (112), although it is much easier to do when $f(r) < r$. The reason why this metric is easy to deal with is of course that it has a very special (diagonal and ϕ-independent form). This means that the surfaces we obtain will be invariant under rotations around the origin.

One more example may be helpful: A torus considered as surface of revolution in flat 3-space. Use cylindrical coordinates on the latter, and parametrize the torus in terms of two angles ϕ and u, the latter occurring in

$$(\rho - a)^2 + z^2 = b^2 \qquad \Leftrightarrow \qquad \rho = a + b\cos u \ , \quad z = b \sin u \ . \tag{128}$$

Clearly $a < b$ and both are positive, otherwise this is not a torus. We arrive at

$$\begin{array}{ll} X = \rho\cos\phi & = (a + b\cos u)\cos\phi \\ Y = \rho\sin\phi & = (a + b\cos u)\sin\phi \\ Z = z & = b\sin u \end{array} \ . \tag{129}$$

A minor calculation shows that

$$ds^2 = b^2 du^2 + (a + b\cos u)^2 d\phi^2 \ , \tag{130}$$

which is well behaved everywhere (since $a < b$). If we set $r = bu$ this is a metric on our standard form, and we get the curvature scalar as

$$R = \frac{2}{b}\frac{\cos\frac{r}{b}}{a + b\cos\frac{r}{b}} = \frac{2}{b}\frac{\cos u}{a + b\cos u} \ . \tag{131}$$

We see that the curvature vanishes if and only if $u = \pm\pi/2$. Also

$$\int_{\rm torus} R = \int_0^{2\pi} du \int_0^{2\pi} d\phi \ \sqrt{g} \ R = 0 \ , \tag{132}$$

where g, by definition, is the determinant of the g_{ij}. This is the same result as for the flat torus. In fact the integral of R over an entire closed two dimensional surface is a topological invariant, independent of the metric.

Finally, a look at the geodesic equation. With only two non-vanishing Christoffel symbols, as in eq. (113), it gives

$$\ddot{r} + \Gamma^r{}_{\phi\phi}\dot{\phi}\dot{\phi} = \ddot{r} - ff'\dot{\phi}\dot{\phi} \tag{133}$$

$$\ddot{\phi} + (\Gamma^\phi{}_{r\phi} + \Gamma^\phi{}_{\phi r})\dot{r}\dot{\phi} = \ddot{\phi} + 2\frac{f'}{f}\dot{r}\dot{\phi} \ . \tag{134}$$

Actually you do not get the solution without a bit of work, if you want the general solution. But if the space we are on is highly symmetric we can simplify things. In particular, consider the sphere ($f = \sin r$), and consider geodesics that start from $r = 0$. Clearly, with these initial data the solutions are

$$r = \sigma \qquad\qquad \phi = \text{constant} \ . \tag{135}$$

As you see, these are Great Circles on the sphere (lines of constant longitude). But on a sphere there is nothing special about the point described in our coordinate system by $r = 0$. By changing coordinates we can place the origin $r = 0$ anywhere. Therefore all Great Circles are geodesics, and conversely. The problem is completely solved for the sphere. Notice the structure of the argument: You have to see through the coordinate system, and think about the real Thing.

Geodesic deviation

It is time to trot out the Riemann tensor for a little useful work. Vaguely speaking, we want to know if geodesics that start out "parallel" to each other eventually converge from each other (as they do on a sphere), or if they diverge. To take the vagueness away from the question, imagine a little vector ξ^α pointing from one geodesic (with tangent vector V^α) to the other, and move it along the geodesic using the covariant derivative along the geodesic, $\nabla_V \equiv V^\alpha\nabla_\alpha$. The "acceleration" of ξ^α is then given by

$$\nabla_V \nabla_V \xi^\alpha = R^\alpha{}_{\mu\nu\beta} V^\mu V^\nu \xi^\beta \ . \tag{136}$$

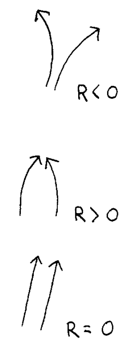
We will derive this equation presently. But we begin by seeing what it says, assuming it is true. In two dimensions we can use eq. (117) to simplify the equation—that goes under the name *geodesic deviation equation*—to

$$\nabla_V \nabla_V \xi^\alpha = \frac{1}{2} R (V^\alpha V \cdot \xi - \xi^\alpha V^2) \ . \tag{137}$$

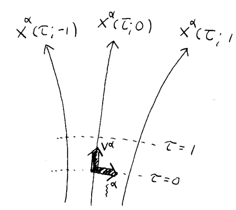
But $V \cdot \xi = 0$ because we choose to define ξ^α that way, and $V^2 = 1$ because we assume that our geodesics are parametrized by arc length. Therefore the geodesic deviation equation in two dimensions has been simplified to

$$\nabla_V \nabla_V \xi^\alpha = -\frac{1}{2} R \xi^\alpha \ . \tag{138}$$

If $R < 0$ the acceleration of the geodesics relative to each other is positive, so that ξ^α will get larger and the geodesics will diverge from each other. If $R > 0$ they converge. In flat space there is no acceleration between straight lines, as you knew perfectly well.



We have arrived at a clearcut meaning of curvature, as something that measures the focussing of geodesics if it is positive, and their dispersion if it is negative. It is easy to check that this fits with what actually happens on spheres and saddles. In dimensions higher than two geodesics may converge in some directions and diverge in others, because the Riemann tensor is no longer encoded in one function only.



Now for a derivation of eq. (136). I will give a slick derivation that may be hard to follow, but—I hope—sufficiently transparent so that you see how the calculations almost do themselves if you know how to handle the formalism. The setup is like this: I have a one-parameter family of geodesics. Let me call them $x^\alpha(\tau; \sigma)$. Here τ is the parameter *along* the geodesics, and σ tells you *which* geodesic that you are on. I will assume that τ and σ can be used as coordinates on a surface in space-time, swept out be the geodesics. This will be true locally at least. Thus

$$\partial_\tau = V^\alpha \partial_\alpha \tag{139}$$

is the tangent vector of a geodesic, and

$$\partial_\sigma = \xi^\alpha \partial_\alpha \tag{140}$$

is a vector connecting nearby geodesics (as it were). I am interested in how ξ^α changes as we move along a given geodesic.

The calculation, keeping eq. (102) and the geodesic equation (101) in mind, starts with the observation that the vector fields commute:

$$[V^\alpha \partial_\alpha, \xi^\beta \partial_\beta] = 0 \ . \tag{141}$$

This implies that

$$V^\beta \partial_\beta \xi^\alpha - \xi^\beta \partial_\beta V^\alpha = V^\beta \nabla_\beta \xi^\alpha - \xi^\beta \nabla_\beta \xi^\alpha = 0 \ . \tag{142}$$

(The affine connection drops out from this particular combination of derivatives.) Then the formalism takes over:

$$\begin{aligned} \nabla_V \nabla_V \xi^\alpha = V^\beta \nabla_\beta (V^\gamma \nabla_\gamma \xi^\alpha) \stackrel{!}{=} V^\beta \nabla_\beta (\xi^\gamma \nabla_\gamma V^\alpha) = \\ = V^\beta \nabla_\beta \xi^\gamma \nabla_\gamma V^\alpha + V^\beta \xi^\gamma \nabla_\beta \nabla_\gamma V^\alpha = \\ = \xi^\beta \nabla_\beta V^\gamma \nabla_\gamma V^\alpha + V^\beta \xi^\gamma \nabla_\gamma \nabla_\beta V^\alpha + V^\beta \xi^\gamma [\nabla_\beta, \nabla_\gamma] V^\alpha = \\ = \xi^\gamma \nabla_\gamma (V^\beta \nabla_\beta V^\alpha) + V^\beta \xi^\gamma R^\alpha{}_{\delta\beta\gamma} V^\delta \ . \end{aligned} \tag{143}$$

The first term is zero because of (101), so

$$\nabla_V \nabla_V \xi^\alpha = R^\alpha{}_{\delta\beta\gamma} V^\delta V^\beta \xi^\gamma \ , \tag{144}$$

that is, we have proved the geodesic deviation equation. And one of my points was that once you know the formalism, lots of calculations almost do themselves.