

CLASSICAL AND QUANTUM THEORY OF PERTURBATIONS
IN INFLATIONARY UNIVERSE MODELS^{*}

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ABSTRACT

A brief introduction to the gauge invariant classical and quantum theory of cosmological perturbations is given. The formalism is applied to inflationary Universe models and yields a consistent and unified description of the generation and evolution of fluctuations. A general formula for the amplitude of cosmological perturbations in inflationary cosmology is derived.

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1. Introduction:

According to the cosmological principle, the Universe should be homogeneous on large scales. The isotropy of the cosmic microwave background temperature to an accuracy of better than 10^{-4} is a powerful confirmation of this principle. As a point of further support, the most recent large-scale redshift surveys¹⁾ indicate a convergence to homogeneity also in the distribution of light.

However, on smaller scales inhomogeneities exist: galaxies, cluster of galaxies, voids and superclusters. The isotropy of the microwave background on smaller scales is an imprint of the homogeneity of the matter distribution at the time of recombination. Hence, it is rather natural to work under the hypothesis that the present structure of the Universe originates from the growth of initially small cosmological perturbations.

At first glance, the theory of linear cosmological perturbations appears straightforward. Given a Friedmann-Robertson-Walker (FRW) background model $(g_{\mu\nu}^{(0)}, T_{\mu\nu}^{(0)})$ and small perturbations $(\delta g_{\mu\nu}, \delta T_{\mu\nu})$ of metric and energy-momentum tensor, we linearize the Einstein equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1}$$

about the background solution to obtain

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu}. \tag{2}$$

The goal of the analysis of these equations is to find the time dependence of the fractional density contrast $\delta\varepsilon/\varepsilon$.

Linear cosmological perturbation theory was first developed by Lifshitz in 1946, but prior to 1980 there was missing motivation for any in depth study, the reason being that there was no causal theory for the origin of fluctuations and hence no reason to study perturbations except on length scales smaller than the Hubble radius where Newtonian theory is adequate.

With the advent of inflationary Universe models, the situation changed drastically. As shown in Fig. 1, provided that the period of inflation is sufficiently long, all scales of cosmological interest originate inside the Hubble radius during the de Sitter phase. Since there is Hawking radiation in the de Sitter space²⁾ with temperature $T_H \sim H$, where H is

Figure 1: Evolution of scales in the inflationary Universe. The comoving wavelength λ of a perturbation is constant in comoving coordinates x_c . The Hubble radius increases after inflation ($t > t_R$), but decreases exponentially during inflation ($t < t_R$).

the expansion rate, fluctuations are produced. These inhomogeneities evolve on scales much larger than the Hubble radius. Hence, a general relativistic analysis is required.

The Hawking radiation argument for the origin of perturbations given above is too naive³⁾. The correct analysis uses the familiar quantum particle production effects for quantum matter fields in an expanding background, as applied to the scalar fields which drive inflation. If quantum fluctuations provide the seed perturbations for structure in the Universe, then a quantum theory of cosmological perturbations is required.

Hence, the key issues within the theory of cosmological perturbations are

- to understand the growth of inhomogeneities on scales larger than the Hubble radius,
- to develop a quantum theory of cosmological perturbations,
- to explain the quantum to classical transition for fluctuations.

In this lecture, we will develop the classical and quantum theory of cosmological perturbations, based on a recent comprehensive review article⁴⁾. The issue of the quantum to classical transition will not be addressed (see e.g. Ref. 5 for literature on this topic). First,

we will demonstrate that the analysis of the perturbation equation (2) is not trivial: there are gauge ambiguities, and the best way to address this problem is to adopt an explicitly gauge invariant formalism.

2. Issues of Gauge:

In a general context, the gauge ambiguity can be described in two ways. In the passive view, we are given a space-time manifold M , a physical quantity Q defined on M , and some corresponding coordinate function ${}^{(0)}Q(x)$ (in the above example, ${}^{(0)}Q(\underline{x}, t) = (t/t_0)^{1/2}$). Let us now choose two sets of coordinates x and \tilde{x} on M . For the first choice, the perturbation $\delta Q(p)$ of Q at a point $p \in M$ is defined as

$$\delta Q(p) = Q(p) - {}^{(0)}Q(x(p)), \quad (3)$$

whereas for the second choice

$$\delta \tilde{Q}(p) = Q(p) - {}^{(0)}Q(\tilde{x}(p)). \quad (4)$$

For small coordinate changes, the transformation $\delta Q(p) \rightarrow \delta \tilde{Q}(p)$ is called a gauge transformation.

In the active view, we are given two manifolds, the space-time manifold M and an unperturbed background manifold N with a fixed coordinate choice. To each physical quantity Q on M there is a corresponding function ${}^{(0)}Q(x)$ on N . Any coordinate choice on M corresponds to some map from N to M (see Fig. 2), and hence to a different definition of $\delta Q(p)$ (see (3) and (4)).

There are two approaches to the gauge problem. One is to fix the gauge, the other is to work in terms of gauge invariant variables. We will now argue that the use of gauge invariant variables has many advantages.

Gravity is not the only theory with gauge ambiguities: electromagnetism is another important example. In electromagnetism we can either work in terms of the gauge dependent potential A_μ or in terms of the gauge invariant field strength tensor $F_{\mu\nu}$. When using A_μ , the homogeneous Maxwell equations are automatically satisfied, and only the inhomogeneous ones need to be solved explicitly. Thus, working in terms of A_μ makes the analysis easier in

Figure 2: The active view of a coordinate transformation: the two mappings \mathcal{D} and $\tilde{\mathcal{D}}$ from the background manifold N to the physical manifold M give rise to two different coordinatizations of M and hence to differing definitions of perturbed quantities.

the sense that less equations must be solved. The disadvantage is that the variables have no direct physical meaning, and that gauge artifacts like the Dirac string may appear.

In gravitational perturbation theory, however, no simplification of the equations is achieved by using gauge dependent variables. Rather, there are more equations and the analysis is more difficult. The interpretational problems remain. Hence, there is strong motivation to adopt the gauge invariant formalism.

There is an additional reason for favoring the gauge invariant approach over working in the usual gauge-synchronous gauge. In synchronous gauge there is a residual gauge freedom which leads to unphysical modes. Although it is in principle possible to subtract these modes, in practice there are formidable difficulties, especially when working with approximate solutions.

Early attempts to develop a gauge invariant theory of cosmological perturbations go back some time⁶⁾. The first completely gauge invariant analysis was achieved by Gerlach and Sengupta⁷⁾ and Bardeen⁸⁾. This lagrangean approach was further developed and clarified in several papers⁹⁾. More recently, an alternative Eulerian (or covariant) analysis has been

developed in Refs. 10 and 11 and in many subsequent papers¹²⁾. The equations of Ref. 8 were rederived in Ref. 13 using the Arnowitt-Deser-Misner approach. For a recent review of the classical and quantum theory of cosmological perturbations, the reader is referred to Ref. 4.

3. Classical Perturbations:

3.1 *Formalism*

There are three types of linear cosmological perturbations: scalar, vector and tensor modes. The names refer to the way in which the modes transform under background space coordinate transformations (see e.g. Ref. 14). Tensor modes are gravitational waves, vector perturbations correspond to rotation and do not grow in time, and only the scalar modes couple (via the Einstein equations) to energy density and pressure. Hence, we shall restrict our attention to scalar type cosmological perturbations.

The first step in the analysis of cosmological perturbations is to identify the gauge invariant combinations of $\delta g_{\mu\nu}$. The general scalar metric perturbation can be written in terms of four scalar functions ϕ , ψ , B and E

$$\delta g_{\mu\nu} = a^2 \begin{pmatrix} 2\phi & -B_{,i} \\ -B_{,i} & 2(\psi\delta_{ij} - E_{,ij}) \end{pmatrix}. \quad (5)$$

For simplicity, we have restricted our attention to the case of a spatially flat background. The following gauge transformations preserve the scalar character of $\delta g_{\mu\nu}$:

$$\begin{aligned} \tilde{\eta} &= \eta + \xi^0 \\ \tilde{x}^i &= x^i + \gamma^{ij}\xi_{,j} \end{aligned} \quad (6)$$

where ξ^0 and ξ are functions of space and time. It is not hard to check that the induced changes of ϕ , ψ , B and E are

$$\begin{aligned} \tilde{\phi} &= \phi - \frac{a'}{a}\xi^0 - \xi^{0'} \\ \tilde{\psi} &= \psi + \frac{a'}{a}\xi^0 \\ \tilde{B} &= B + \xi^0 - \xi' \\ \tilde{E} &= E - \xi, \end{aligned} \quad (7)$$

where a prime denotes the derivative with respect to conformal time. Now it is a simple

exercise in linear algebra to find a basis of gauge invariant variables. A convenient choice is

$$\begin{aligned}\Phi &= \phi + a^{-1}[(B - E')a]' \\ \Psi &= \psi - \frac{a'}{a}(B - E').\end{aligned}\tag{8}$$

Note that in longitudinal gauge ($B = E = 0$) the gauge invariant variables become $\Phi = \phi$ and $\Psi = \psi$.

The second step of our analysis is to derive the equations of motion for the gauge invariant variables. In principle, this is straightforward. The linearized Einstein equations (2) are conveniently combined to yield equations for Φ and Ψ . In practice, this computation is rather tedious unless a clever procedure is chosen. It is simplest⁴⁾ to consider first the transformation of the perturbation $\delta G_{\mu\nu}$ of the Einstein tensor under (6), and to determine gauge invariant combinations (labelled with superscript (gi)):

$$\begin{aligned}\delta G_0^{0(gi)} &= \delta G_0^0 + {}^{(0)}G_0^{0'}(B - E') \\ \delta G_i^{0(gi)} &= \delta G_i^0 + \left({}^{(0)}G_0^0 - \frac{1}{3} {}^{(0)}G_k^k \right) (B - E')_{,i} \\ \delta G_j^{i(gi)} &= \delta G_j^i + {}^{(0)}G_j^{i'}(B - E')\end{aligned}\tag{9}$$

where the background Einstein tensor elements are ${}^{(0)}G_\nu^\mu$. Evidently, the analogous combinations of δT_ν^μ are gauge invariant. Thus, the linearized Einstein equations can be written as

$$\delta G_\nu^\mu(gi) = 8\pi G \delta T_\nu^\mu(gi).\tag{10}$$

In this form, all the gauge dependence automatically drops out, and we obtain the following set of equations written exclusively in terms of gauge invariant variables:

$$\begin{aligned}-3\mathcal{H}(\mathcal{H}\Phi + \Psi') + \nabla^2\Psi &= 4\pi G a^2 \delta T_0^{0(gi)} \\ (\mathcal{H}\Phi + \Psi')_{,i} &= 4\pi G a^2 \delta T_i^{0(gi)} \\ \left[(2\mathcal{H}' + \mathcal{H}^2)\Phi + \mathcal{H}\Phi' + \Psi'' + 2\mathcal{H}\Psi' + \frac{1}{2}\nabla^2 D \right] \delta_j^i - \frac{1}{2}\gamma^{ik} D_{,kj} & \\ &= -4\pi G a^2 \delta T_j^{i(gi)},\end{aligned}\tag{11}$$

where $D = \Phi - \Psi$ and $\mathcal{H} = a'/a$.

An alternative way to derive the above equations⁴⁾ is to work in longitudinal gauge and at the end replace ϕ and ψ by Φ and Ψ respectively (and similarly for the matter variables).

3.2 Applications

As a first application of the classical theory of cosmological perturbations we shall consider the example of perfect fluid matter given by the energy-momentum tensor

$$T_{\beta}^{\alpha} = (\varepsilon + p)u^{\alpha}u_{\beta} - p\delta_{\beta}^{\alpha}, \quad (12)$$

ε and p being energy density and pressure respectively, and u^{α} the four velocity vector of the fluid. In general, the pressure is a function of both ε and entropy per baryon s , and hence

$$\delta p = c_s^2 \delta \varepsilon + \tau \delta s, \quad (13)$$

with c_s being the speed of sound. If $\tau = 0$, we have a pure adiabatic perturbation.

The perturbation of T_{β}^{α} is given by

$$\begin{aligned} \delta T_0^0 &= \delta \varepsilon \\ \delta T_i^0 &= (\varepsilon_0 + p_0)a^{-1}\delta u_i \\ \delta T_j^i &= -\delta p\delta_j^i, \end{aligned} \quad (14)$$

where subscripts denote background quantities. Since δT_j^i is diagonal, it follows immediately from the third equation in (11) that $\Phi = \Psi$. This, in turn, leads to a significant simplification of the equations of motion for the gauge invariant variables. From the first equation in (11), we obtain

$$\nabla^2 \Phi - 3\mathcal{H}\Phi' - 3\mathcal{H}^2\Phi = 4\pi G a^2 \delta\varepsilon^{(gi)}. \quad (15)$$

This is a generalization of the Poisson equation to which it reduces in the Newtonian limit; and hence we call Φ the relativistic potential.

Equations (11) can be combined to yield the following second order equation of motion for Φ :

$$\Phi'' + 3\mathcal{H}(1 + c_s^2)\Phi' - c_s^2\nabla^2\Phi + [2\mathcal{H}' + (1 + 3c_s^2)\mathcal{H}^2]\Phi = 4\pi G a^2 \tau \delta s \quad (16)$$

For adiabatic perturbations, the source term vanishes. On scales larger than the Hubble radius, the spatial gradients can be neglected. Under these conditions, equation (16) can be

recast as a “conservation law”

$$\dot{\zeta} = 0 \quad (17)$$

where the dot denotes the derivative with respect to physical time and

$$\zeta = \frac{2}{3} \frac{H^{-1}\dot{\Phi} + \Phi}{1+w} + \Phi \quad (18)$$

with $w = p/\varepsilon$. The quantity ζ was first introduced in Ref. 15 (see also Ref. 16). The above conservation law is easily applicable to many interesting issues. First, we note that if the equation of state is constant, then Φ remains constant (the second solution of (17) is a decaying mode). However, during a phase transition w may change by a large factor. In this case, equation (17) implies that the relativistic potential Φ will also change by a large factor. This is one of the key points in the computation of density perturbations from inflation¹⁷⁾.

To correctly describe fluctuations from inflation, we must consider a second application of the classical theory of cosmological perturbations, namely a model with scalar field matter. The matter action is

$$S_m = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} \varphi^{;\alpha} \varphi_{;\alpha} - V(\varphi) \right\}, \quad (19)$$

semicolons denoting the covariant derivative. The induced energy- momentum tensor of the scalar field φ is

$$T_{\beta}^{\alpha} = \varphi^{;\alpha} \varphi_{;\beta} - \left\{ \frac{1}{2} \varphi^{;\gamma} \varphi_{;\gamma} - V(\varphi) \right\} \delta_{\beta}^{\alpha}. \quad (20)$$

If we expand $\varphi(\underline{x}, t)$ about a homogeneous background field $\varphi_0(t)$

$$\varphi(\underline{x}, t) = \varphi_0(t) + \delta\varphi(\underline{x}, t), \quad (21)$$

then the perturbation of T_{β}^{α} at the linearized level becomes

$$\begin{aligned} \delta T_0^0 &= a^{-2} \{ -\varphi_0'^2 \phi + \varphi_0' \delta\varphi' + V_{,\varphi} a^2 \delta\varphi \} \\ \delta T_i^0 &= a^{-2} \varphi_0' \delta\varphi_{,i} \\ \delta T_j^i &= a^{-2} \{ \varphi_0'^2 \phi - \varphi_0' \delta\varphi' + V_{,\varphi} a^2 \delta\varphi \} \delta_j^i. \end{aligned} \quad (22)$$

As in the case of a perfect fluid, δT_j^i is diagonal and hence $\Phi = \Psi$

Inserting this result and (22) into the general equations (11) and combining the resulting differential equations, we obtain the following second order equation for Φ

$$\Phi'' + 2 \left(\mathcal{H} - \frac{\varphi_0''}{\varphi_0'} \right) \Phi' - \nabla^2 \Phi + 2 \left(\mathcal{H}' - \frac{\varphi_0''}{\varphi_0'} \mathcal{H} \right) \Phi = 0. \quad (23)$$

Since for a scalar field

$$1 + w = \frac{\dot{\varphi}_0^2}{\varepsilon}, \quad (24)$$

we can, like for perfect fluid matter, rewrite (23) as a ‘‘conservation law’’ identical to (17) and (18) when considering scales much larger than the Hubble radius.

3.3 Fluctuations in Inflationary Cosmology

To demonstrate how easy it is to apply the gauge invariant theory of cosmology perturbations, we shall consider the evolution of fluctuations in inflationary Universe models^{15–19}. We first note from (11) that on scales smaller than the Hubble radius

$$\Phi = -\frac{3}{2} \left(\frac{aH}{k} \right)^2 \left(\frac{\delta\varepsilon}{\varepsilon} \right)^{(gi)}. \quad (25)$$

The calculation of density perturbations proceeds as follows: by evaluating (25) at the time $t_i(k)$ (see Fig. 1) when the wavelength under consideration leaves the Hubble radius, we determine the initial value of Φ , $\Phi(t_i(k))$. By integrating (17) and using the fact that $\dot{\Phi}$ vanishes at both $t_i(k)$ and $t_f(k)$, the value of Φ at the time $t_f(k)$ when the scale reenters the Hubble radius can be calculated with the result

$$\Phi(t_f(k)) = \frac{1 + w(t_f)}{\frac{5}{3} + w(t_f)} \frac{2}{3} \frac{\Phi(t_i)}{1 + w(t_i)} \equiv \alpha \frac{\Phi(t_i)}{1 + w(t_i)}. \quad (26)$$

The coefficient α is 4/9 for t_f in the radiation dominated phase and $\alpha = 2/5$ during matter domination. Using (25), the value of $\Phi(t_f)$ determines the late time value of the amplitude of the fractional density perturbation.

In order to evaluate the amplitude of perturbations (26) in inflationary Universe models, a quantum analysis of the generation of fluctuations is required. However, already a rough

order of magnitude estimate yields interesting results. From (25) and taking the energy density perturbation to be given by the Hawking temperature (i.e. $\delta\varepsilon \sim H^4$) we obtain

$$\Phi(t_i) \sim \frac{H^4}{\varepsilon} \ll 1. \quad (27)$$

Inflation is driven by a scalar field φ . During inflation, the equation of state is dominated by the potential energy density of φ . However, φ is rolling and therefore

$$1 + w(t_i) = \frac{\dot{\varphi}^2}{\varepsilon}. \quad (28)$$

On dimensional grounds, $\dot{\varphi}^2 \sim H^4$ and hence $\Phi(t_f) \sim 1$. We conclude that the change in the equation of state leads to a drastic amplification of the initial quantum fluctuations. Successful models of galaxy formation require $\Phi(t_f) \sim 10^{-4}$. Thus, without careful adjustment of parameters, inflationary Universe models predict too large perturbations¹⁸⁾.

4. Quantum Perturbations:

4.1 *Motivation*

The classical analysis of fluctuations in inflationary Universe models gives good insight into why initially tiny inhomogeneities are amplified by a large factor between when they are produced in the de Sitter phase and when they reenter the Hubble radius at late times. It is, however, only a quantum analysis which explains the origin of these perturbations. It is vacuum quantum fluctuations which are the source of the classical inhomogeneities which form the seeds for galaxy and cluster formation.

A second motivation for considering the quantum theory of cosmological perturbations comes from the general problem of particle production in expanding background spacetimes. The usual²⁰⁾ treatment which is based on quantizing matter fields on an unperturbed cosmological background is inconsistent since matter fluctuations are intrinsically coupled to metric perturbations via the Einstein equations. Hence, we need to quantize metric and matter fluctuations in a unified way.

In fact, the quantization of linear cosmological fluctuations is not more complicated than the well known quantization of matter fields in an external background: it is a straightforward application of canonical quantization²⁰⁾. In synchronous gauge, the quantization of fluctuations was first discussed in Ref. 17.

Since we only wish to quantize the physical degrees of freedom, it is advantageous to use the gauge-invariant formalism. Since this method reduces the number of degrees of freedom, it also leads to a substantial simplification of the analysis.

The first step in deriving the quantum theory of cosmological perturbations^{21,22)} is to determine the action for the fluctuations in terms of the gauge invariant variables. In general, it would be wrong to simply start from the classical equation of motion for perturbations and interpret it directly as an operator equation. This would lead to wrong canonical momenta and to a wrong normalization of the field operator^{4,23)}.

4.2 *Formalism*

In the following, we shall briefly summarize the quantum theory of cosmological perturbations. For simplicity, only models with scalar field matter will be considered. For hydrodynamical matter the analysis is similar⁴⁾. The formalism also applies to higher derivative

gravity theories^{4,24}). We will follow the method of Ref. 21 (see Ref. 4 for more details).

The first and most involved step in quantizing cosmological perturbations is to write the action for fluctuations in terms of gauge invariant variables only. We start from the action

$$S = \int d^4x \sqrt{-g} R + S_m, \quad (29)$$

where S_m is the action for the scalar field φ . Next, we insert into (29) the expansion of $g_{\mu\nu}$ and φ about a homogeneous background solution $g_{\mu\nu}^{(0)}$ and φ_0

$$\begin{aligned} g_{\mu\nu} &= g_{\mu\nu}^{(0)} + \delta g_{\mu\nu} \\ \varphi &= \varphi_0 + \delta\varphi \end{aligned} \quad (30)$$

and expand the result in terms of powers of small quantities to find

$$S = S_0 + \delta_2 S \quad (31)$$

where S_0 is the action of the background solution and $\delta_2 S$ is quadratic in perturbation variables (the linear terms vanish because we are expanding about a solution of the equations of motion). We now use the constraint equations to simplify the action and drop total derivative terms. After a significant amount of algebra one obtains the following very simple form of $\delta_2 S$:

$$\delta_2 S = \frac{1}{2} \int d^4x \left\{ v'^2 - v_{,i} v_{,j} \delta^{ij} + \frac{z''}{z} v^2 \right\} \quad (32)$$

where v is a gauge invariant combination of matter and metric perturbations

$$v = a \left(\delta\varphi^{(gi)} + \frac{\varphi'_0}{\mathcal{H}} \Phi \right) \quad (33)$$

and

$$z = \frac{a\varphi'_0}{\mathcal{H}}. \quad (34)$$

The result (32) has the same form as the action of a simple scalar field with time dependent square mass $-z''/z$. Note that although the details of the reduction of the action are

somewhat involved, the final result is no surprise. We have seen in section 3 that for scalar field matter there is only one independent gauge invariant metric perturbation variable. Via the Einstein equations this variable is coupled to the gauge invariant matter fluctuations. Thus, this is only one independent variable which expresses in a unified manner both matter and metric perturbations.

From this point on, the quantization prescription is straightforward canonical quantization. From $\delta_2 S$ we can immediately write down the canonical momenta. After imposing the canonical commutation relations, we expand the operator \hat{v} corresponding to the classical field v in terms of creation and annihilation operators a_k^+ and a_k^- :

$$\hat{v} = \frac{1}{2} \frac{1}{(2\pi)^{3/2}} \int d^3k \left[e^{ikx} v_k^*(\eta) a_k^- + e^{-ikx} v_k(\eta) a_k^+ \right]. \quad (35)$$

The mode functions $v_k(\eta)$ satisfy the equation

$$v_k'' + \left(k^2 - \frac{z''}{z} \right) v_k = 0. \quad (36)$$

Since (36) is a harmonic oscillator equation with time dependent mass, there will be quantum particle production²⁰). Modes of (36) which have positive frequency at some initial time t_0 are no longer pure positive frequency at a later time $t_1 > t_0$. This leads to time dependence of expectation values of physical operators. For example, if $|\psi_0\rangle$ is the vacuum state at time t_0 , and $N_k(t_1) = a_k^+(t_1) a_k^-(t_1)$ is the number operator at time t_1 defined in terms of the operator coefficients of the positive frequency modes at time t_1 , then

$$\langle \psi_0 | N_k(t_1) | \psi_0 \rangle \neq 0. \quad (37)$$

The final step is to compute the expectation values of the operators which determine the r.m.s. mass fluctuation. If $\delta M/M(k)$ is the r.m.s. mass perturbation inside a sphere of radius k^{-1} , then

$$\left(\frac{\delta M}{M} \right)^2(k) \sim k^3 \left(\frac{\delta \varepsilon}{\varepsilon} \right)^2(k)^{(gi)} = k^3 \frac{\langle \psi_0 | |\delta \varepsilon^{(gi)}(k)|^2 | \psi_0 \rangle}{\varepsilon^2}, \quad (38)$$

where in the final step we have replaced the classical perturbation by the expectation value of the quantum operator evaluated in the vacuum state $|\psi_0\rangle$ at the beginning of inflation.

This prescription for taking the quantum to classical transition has been discussed in Ref. 5 and references therein. When evaluated at the time of Hubble radius crossing $t_f(k)$ (see Fig. 1), then using the relationship (25) between $\delta\varepsilon^{(gi)}$ and Φ one obtains

$$\left(\frac{\delta M}{M}\right)^2(k, t_f(k)) \sim k^3 \langle \psi_0 | |\Phi(k)|^2 | \psi_0 \rangle . \quad (39)$$

In turn, the gauge invariant potential Φ is related to the variable v by

$$k^2 \Phi = -4\pi G \frac{\varphi_0'^2}{\mathcal{H}} \left(\frac{v}{z}\right)' . \quad (40)$$

Hence, the computation of the expectation value in (39) reduces to a straightforward evaluation of the expectation value of v^2 .

Combining (39) and (40), we find

$$\left(\frac{\delta M}{M}\right)^2(k, t_f(k)) \sim \frac{1}{4\pi^2} \frac{\varphi_0'^2}{a^2} k^3 |u_k(t_f(k))|^2 \quad (41)$$

where $u_k(\eta)$ are proportional to the expansion coefficients of the operator $\hat{\Phi}$:

$$\hat{\Phi}(\underline{x}, \eta) = \frac{1}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \frac{\varphi_0'}{a} \int d^3k \left[u_k^*(\eta) e^{i\mathbf{k}\cdot\underline{x}} a_k^- + u_k(\eta) e^{-i\mathbf{k}\cdot\underline{x}} a_k^+ \right] . \quad (42)$$

Equation (41) relates the r.m.s. mass perturbation resulting from quantum vacuum fluctuations to the solution $u_k(\eta)$ of the classical equation of motion. At this point, we have established a consistent unified treatment of generation and evolution of cosmological perturbations.

Evaluating (41) for a model of chaotic inflation²⁵⁾ with potential

$$V(\varphi) = \frac{\lambda}{n} \varphi^n \quad (43)$$

yields

$$\frac{\delta M}{M}(k, t_f(k)) \sim \frac{\lambda^{1/2}}{m_{pl}} \left(\frac{2n}{3l^2}\right)^{n/4-1/2} \left[l n \left(\frac{k_\gamma}{k}\right)\right]^{n/4+1/2}, \quad (44)$$

where l is the Planck length and k_γ is the characteristic wavenumber of the cosmic microwave background. For $n = 2$ (and setting $\lambda = m^2$), a correct value of $\frac{\delta M}{M}$ for galaxy formation

requires $m \sim 10^{-6}m_{pl}$, for $n = 4$ the requirement is $\lambda \sim 10^{-12}$. As mentioned at the end of Section 3, inflation thus requires careful adjustment of parameters if it is to give the required value of density perturbations.

This completes our brief survey of the quantum theory of cosmological perturbations. Although the formalism has been developed for scalar field matter (and in particular applied to quantum fluctuations in inflationary Universe models), it is much more general. Also for hydrodynamical matter and in higher derivative gravity theories, the action for perturbations can be reduced to a form like (32), and the quantization then proceeds as in the example discussed⁴⁾.

5. Conclusions and Discussion:

We have summarized the gauge invariant theory of classical and quantum cosmological perturbations. It allows a consistent unified treatment of the generation and evolution of linearized fluctuations in inflationary Universe models.

In Section 2 we argued that a gauge invariant analysis of classical perturbations is physically unambiguous and technically straightforward. It eliminates the gauge ambiguities associated with gauge-dependent approaches. The coordinate approach presented here is probably the most simple way of deriving the equations of motion for the gauge invariant gravitational potential. It is action based and hence allows standard canonical quantization. A gauge invariant analysis of the quantum theory implies that only physical degrees of freedom are quantized.

The formalism presented here is practical and can easily be applied to problems of real cosmological interest. Already the classical theory leads to a useful “conservation law” (see (17) and (18)) which allows us to track the amplitude of perturbations on scales much larger than the Hubble radius in a very simple manner.

The quantum theory of cosmological perturbations relates the expectation values of two-point functions which determine the r.m.s. mass fluctuations to mode functions which obey the classical equations of motion for the gauge invariant gravitational potential. This allows a unified analysis of the generation and evolution of density perturbations in inflationary Universe models (see (41)). A general formula for the amplitude of the resulting fluctuations is given.

In Ref. 4 we have performed detailed calculations of the spectrum of density perturbations in models with scalar field matter, hydrodynamical matter, and in higher derivative theories of gravity. The formalism also allows a discussion of entropy perturbations, it can be used to yield a simple proportionality between the microwave background temperature anisotropies and the gravitational potential Φ , and it can be applied to the generation and evolution of gravitational waves.

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REFERENCES

1. S. Shechtman et al., ‘Strip-Mining the Southern Sky: Skratching the Surface’, CFA preprint No. 3385 (1992).
2. E. Lifshitz, *Zh. Eksp. Teor. Fiz.* **16**, 587 (1946).
3. See e.g. discussion session on inflation at ICGC-87, in “Highlights in Gravitation and Cosmology”, ed. by B. Iyer, et. al. (Cambridge Univ. Press, Cambridge, 1988).
4. V. Mukhanov, H. Feldman and R. Brandenberger, *Phys. Rep.* **215**, 203 (1992).
5. M. Sakagami, *Prog. Theor. Phys.* **79**, 443 (1988);
R. Brandenberger, R. Laflamme and M. Mijic, *Phys. Scripta* **T36**, 265 (1991).
6. S. Hawking, *Ap. J.* **145**, 544 (1966);
D. Olson, *Phys. Rev.* **D14**, 327 (1976).
7. U. Gerlach and U. Sengupta, *Phys. Rev.* **D18**, 1789 (1978).
8. J. Bardeen, *Phys. Rev.* **D22**, 1882 (1980).
9. R. Brandenberger, R. Kahn and W. Press, *Phys. Rev.* **D28**, 1809 (1983);
H. Kodama and M. Sasaki, *Prog. Theor. Phys. Suppl.* **78**, 1 (1984);
G. Chibisov and V. Mukhanov, “Theory of Relativistic Potential: Cosmological Perturbations”, Preprint No. 154 of P.N. Lebedev Physical Institute (1983).

10. D. Lyth and M. Mukherjee, *Phys. Rev.* **D38**, 485 (1988).
11. G. Ellis and M. Bruni, *Phys. Rev.* **D40**, 1804 (1989).
12. G. Ellis, J. Hwang and M. Bruni, *Phys. Rev.* **D40**, 1919 (1989);
J. Hwang and E. Vishniac, *Ap. J.* **353**, 1 (1990);
G. Ellis, M. Bruni and J. Hwang, *Phys. Rev.* **D42**, 1035 (1990).
13. R. Durrer and N. Straumann, *Helv. Phys. Acta* **61**, 1027 (1988).
14. J. Stewart, *Class. Quant. Grav.* **7**, 1169 (1990).
15. J. Bardeen, P. Steinhardt and M. Turner, *Phys. Rev.* **D28**, 679 (1983).
16. R. Brandenberger and R. Kahn, *Phys. Rev.* **D29**, 2175 (1984);
D. Lyth, *Phys. Rev.* **D31**, 1792 (1985).
17. G. Chibisov and V. Mukhanov, "Galaxy Formation and Phonons", Lebedev Physical
Institute Preprint No. 162 (1980);
G. Chibisov and V. Mukhanov, *Mon. Not. R. astron. Soc.* **200**, 535 (1982);
V. Lukash, *Pisma Zh. Eksp. Teor. Fiz.* **31**, 631 (1980);
V. Lukash, *Zh. Eksp. Teor. Fiz.* **79**, 1601 (1980).
18. S. Hawking, *Phys. Lett.* **115B**, 295 (1982);
A. Starobinsky, *Phys. Lett.* **117B**, 175 (1982);
A. Guth and S.- Y. Pi, *Phys. Rev. Lett.* **49**, 1110 (1982).
19. V. Mukhanov, *Pisma Zh. Eksp. Teor. Fiz.* **41**, 402 (1985).
20. N. Birrell and P. Davies, "Quantum Fields in Curved Space", (Cambridge University
Press, Cambridge, 1982).
21. V. Mukhanov, *Zh. Eksp. Teor. Fiz.* **94**, 1 (1988).
22. M. Sasaki, *Prog. Theor. Phys.* **76**, 1036 (1986).
23. N. Deruelle, C. Gundlach and D. Polarski, *Class. Quant. Grav.* **9**, 137 (1992).
24. V. Mukhanov, *Phys. Lett.* **218B**, 17 (1989).
25. A. Linde, *Phys. Lett.* **129B**, 177 (1983).